

Optimal Constellations for the Low-SNR Noncoherent MIMO Block Rayleigh-Fading Channel

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Abstract—Reliable communication over the discrete-input/continuous-output noncoherent multiple-input multiple-output (MIMO) Rayleigh block-fading channel is considered when the signal-to-noise ratio (SNR) per degree of freedom is low. Two key problems are posed and solved to obtain the optimum discrete input. In both problems, the average and peak power per space-time slot of the input constellation are constrained. In the first one, the peak power to average power ratio (PPAPR) of the input constellation is held fixed, while in the second problem, the peak power is fixed independently of the average power. In the first *PPAPR-constrained* problem, the mutual information, which grows as $O(\text{SNR}^2)$, is maximized up to second order in SNR. In the second *peak-constrained* problem, where the mutual information behaves as $O(\text{SNR})$, the structure of constellations that are optimal up to first order, or equivalently, that minimize energy per bit, are explicitly characterized. Furthermore, among constellations that are first-order optimal, those that maximize the mutual information up to second order, or equivalently, the wideband slope, are characterized. In both PPAPR-constrained and peak-constrained problems, the optimal constellations are obtained in closed form as solutions to nonconvex optimizations, and interestingly, they are found to be identical. Due to its special structure, the common solution is referred to as space-time orthogonal rank one modulation, or STORM. In both problems, it is seen that STORM provides a sharp characterization of the behavior of noncoherent MIMO capacity.

Index Terms—Capacity, constellation design, energy per bit, low signal-to-noise ratio (SNR), multiple-input multiple-output (MIMO), nonconvex optimization, noncoherent communication, peak power, peak-to-average power ratio, Rayleigh fading, Space-Time Orthogonal Rank one Modulation (STORM), wideband slope.

I. INTRODUCTION

IN this paper, we consider the problem of communicating reliably over a multiple-input multiple-output (MIMO) block Rayleigh-fading channel in the low signal-to-noise ratio (SNR) regime. We assume the noncoherent model, wherein neither the transmitter nor the receiver are assumed to have instantaneous channel state information (CSI), while both have knowledge of the channel distribution. In scenarios where

the mobile receivers are moving at a high speed or when the number of transmit antennas is large, channel estimation at the receiver might be insufficient due to the small coherence times involved. The problem of the receiver acquiring CSI is further exacerbated in the low-SNR regime, where the channel estimates can be unreliable. As a result, the more common assumption of perfect CSI at the receiver, namely, that of coherent communications, may not hold true in such cases.

A more fundamental rationale for studying the noncoherent model is as follows. Since in practice the channel is not known to the receiver at the start of communication, an information-theoretic formulation of the noncoherent problem—which implicitly accounts for the resources needed for implicit channel estimation without constraining the transmission scheme in any way—is more fundamental than the coherent formulation. Systems that assume coherent transmission by arguing that the channel can be acquired at the receiver by the use of pilot-symbol-assisted transmission to perform explicit channel estimation are inherently suboptimal in general while not taking into account the resources, namely, energy and degrees of freedom, needed for pilot transmissions, as they should.

The study of noncoherent fading channels at low SNR is motivated by their application in wideband (WB) and ultra-wideband (UWB) channels. In such scenarios, the signal power is spread over a large bandwidth, rendering the SNR per degree of freedom low. Transmissions over WB fading channels experience both time and frequency selectivity. However, within a short window of time or frequency, the channel fading coefficients are known to be highly correlated. One widespread approach therefore to deal with frequency selectivity is to divide the original wideband channel into several parallel narrowband channels such that each narrowband channel experiences flat fading or a single tap coefficient. To deal with time selectivity, a common approach is to model each narrowband channel through *block fading*. In the block-fading model, the channel coefficients are assumed fixed for a duration in time following which they assume independent and identically distributed (i.i.d.) realizations (here adequate interleaving across time and frequency windows is implicitly assumed). In this work, we model the wideband channel as a block-faded narrowband channel in the low-SNR regime. This simplifying channel modeling assumption helps capture the essence of the original wideband channel, and is widely adopted in the analysis of MIMO fading channels.

The study of noncoherent single-input single-output (SISO) fading channels at low SNR dates back to the 1960s. Two equivalent notions of optimality in the literature that are indicators of

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energy efficiency in the low-SNR regime are 1) the input being first-order optimal with respect to Shannon capacity or 2) the input achieving the minimum energy per bit or $\frac{E_b}{N_0 \min}$ required for reliable communication. A classical result by Shannon [1] is that in the limit of infinite bandwidth or vanishing SNR, the minimum energy per bit required for reliable communications over an additive white Gaussian noise (AWGN) channel is -1.59 dB. Early work by Kennedy [2], Jacobs [3] (also see Gallager [4] and the references therein) studied wideband SISO Rayleigh-fading channels with an average power constrained input and showed that in the limit of infinite bandwidth or vanishing SNR, the required minimum energy per bit is again -1.59 dB, the same as that of an AWGN channel. A remarkable observation then was that the minimum energy per bit required is the same whether or not the receiver has knowledge of the channel fading coefficients. Telatar and Tse [5], and Verdú [6] show that the minimum energy per bit is -1.59 dB even for fairly general multipath SISO fading channels and general MIMO fading channels, respectively. A common approach adopted to obtain $\frac{E_b}{N_0 \min}$ for fading channels is to consider the achievable rate of a certain scheme (often M -ary frequency-shift keying or MFSK), which is transmitted at arbitrarily low duty cycles (cf. [2], [4], [5]). The required result is then obtained by either showing that the energy per bit of the scheme at vanishing SNR matches that of the AWGN channel, or by deriving an upper bound on capacity that is tight with respect to the achievable lower bound. However, this approach fixes the input *a priori*, and therefore no determination can be made as to the *necessary* conditions for a constellation to achieve the minimum energy per bit. The characterization of the class of signals (more generally, input distributions) that are both necessary and sufficient to achieve the minimum energy per bit had been an important and long-standing open problem.

Signals such as arbitrarily low duty-cycled FSK tend to have prohibitively large peak-to-average-power ratios (PAPR) and are consequently difficult to implement in practice. Such signals are therefore referred to as “peaky” signals in the literature. Using certain types of fourth moments of the input as measures of peakiness, Médard and Gallager [7], and Subramanian and Hajek [8] showed that signaling that is not peaky in either time or frequency dimensions cannot achieve the minimum energy per bit as $\text{SNR} \rightarrow 0$. Verdú [6] formalized this notion further for fairly general noncoherent MIMO fading channels and established that *flash signaling*, where the input distribution converges to a zero mass and a nonzero mass that is transmitted with vanishing probability as $\text{SNR} \rightarrow 0$, is both necessary and sufficient to achieve the minimum energy per bit. While noncoherent communications is sufficient to transmit at the AWGN minimum energy per bit of -1.59 dB, the work in [6] resolves another major difficulty. It introduces and explains the crucial role of *wideband slope* (\mathcal{S}_0) at large but finite bandwidths. The wideband slope is a measure of how fast the energy per bit of the optimal scheme approaches the minimum energy per bit, and is synonymous with the notion of second-order optimality with respect to Shannon capacity. One main result of [6] is that for noncoherent MIMO channels with an average power constraint, the wideband slope is zero. This result implies that to approach the minimum energy per bit, the bandwidth for reliable nonco-

herent communications becomes prohibitively large and the associated signaling scheme prohibitively peaky, and therefore no realistic (i.e., bandwidth-limited and peak-limited) scheme can achieve the minimum energy per bit.

Hence, it was important to pose problems that provide meaningful second-order performance when considering noncoherent fading channels at low SNR. One way was to impose suitable peak constraints on the input. It is shown in Rao and Hassibi [9] that under certain regularity conditions on the signal, which include making the fourth and sixth moments finite, the noncoherent MIMO capacity grows as $O(\text{SNR}^2)$. Similar expressions for the mutual information up to the second order are obtained in closed form in [10], [11] with different assumptions on the fading matrices and peak-power constraints. Even though such problems have capacity behaving as $O(\text{SNR}^2)$, and hence the minimum energy per bit not occurring at a vanishing SNR, they are important since they involve practical modulation schemes with reasonable PAPR. Schemes designed to satisfy such regularity conditions must be deployed in the vicinity of the SNR where the minimum energy per bit is achieved. Also relevant is the interesting case of the peak-constraint imposed being independent of the average power constraint, resulting in $O(\text{SNR})$ growth of capacity. In this case, it will be shown here that the wideband slope is not zero anymore (unlike the average power constraint only problem). Therefore, the energy per bit approaches the minimum energy per bit at a nonzero rate as $\text{SNR} \rightarrow 0$. Gursoy and Verdú [12] consider SISO Rician fast-fading channels and impose different peak power constraints in addition to the average power constraint on the input. For certain combinations of peak and average-power constraints, they characterize the $\frac{E_b}{N_0 \min}$ and \mathcal{S}_0 for SISO Rician fast-fading channels. For a combination of peak and average power constraints, they show that on-off quadrature phase shift keying (OOQPSK) achieves the minimum energy per bit as well as the optimal wideband slope for the noncoherent SISO Rician fast-fading channel. This result is obtained in [12] by directly evaluating a second-order expansion of mutual information for OOQPSK, and this approach cannot be extended to more general MIMO block-fading models. To the best of the authors’ knowledge, this is the only input distribution reported in the literature that is both first- and second-order optimal, in the context of peak-constrained noncoherent communications over fading channels.

Abou-Faycal *et al.* [13] consider a noncoherent SISO Rayleigh fast-fading channel and prove that the capacity achieving distribution is discrete with a finite number of points, one of them being at the origin. In [14], the authors consider a SISO Rician fast-fading channel and show that the capacity-achieving distribution is discrete even when certain types of peak-constraints are imposed. While there is no formal proof of the discreteness of the optimal input for MIMO Rayleigh-fading channels, it is expected to be the case. Despite these results, discrete input optimization of information-theoretic measures is rarely considered since the optimizations encountered are often seen as being analytically intractable. Another compelling reason for considering the problem of maximizing mutual information as a finite-dimensional optimization, over a discrete and finite cardinality input is that, the

solution, if obtainable, would offer insights simultaneously into information-theoretic as well as coding-modulation aspects. For, consider that even when capacity achieving probability distribution functions are found, the problem of practical transmission would be still unresolved as it would not be clear how the choice of a quantization of the optimum input would affect performance. Some recent works that deal with discrete signal constellation design using information-theoretic criteria but only under an average power constraint and via *numerical* optimization techniques are [15]–[17]. While the results in [16] provided numerically computable tight lower bounds on capacity of the noncoherent MIMO channel, the associated constellations may be hard to implement in practice due to their limited analytical structure and lack of strict peak or peak-to-average power ratio constraints.

In this paper, we pose and solve two key problems of obtaining the optimum discrete input of finite cardinality for peak-constrained MIMO noncoherent block Rayleigh-fading channels in closed form. Given the results of [13], [14], it is expected that there will be no loss in optimizing over discrete inputs as opposed to input distribution functions. In both problems, we assume average power constraints on the input. In addition, we also assume natural peak constraints per antenna and per time slot, which closely emulate constraints on power amplifiers, instead of fourth- and higher order moment constraints on the input used in [7]–[9]. In the first problem, the peak-power to average-power ratio (PPAPR) of the input constellation is held fixed, while in the second, the peak power is fixed independently of the average power. We refer to these two problems as the *PPAPR-constrained* and *peak-constrained* cases, respectively. We show that interestingly, in the case of the noncoherent MIMO Rayleigh-fading channel at low SNR, such joint optimizations of information-theoretic metrics over complex signal matrices and their respective probabilities are indeed analytically tractable and result in elegant closed-form solutions.

In the PPAPR constrained case, it can be shown that the input satisfies certain regularity conditions specified in [9]. For such inputs, the mutual information is obtained up to second order in [9] and shown to grow as $O(\text{SNR}^2)$. In one of the key contributions here, we maximize this second-order mutual information jointly over the matrix-valued elements of a finite input constellation and their probabilities, when the cardinality of the constellations is no greater than $T + 1$, where T is the channel coherence block length.

In the peak-constrained case, the mutual information behaves as $O(\text{SNR})$. Here, we explicitly characterize the structure of constellations of any finite cardinality that are optimal up to first order, or equivalently, that minimize energy per bit or maximize capacity per unit energy. More importantly, among constellations of cardinality no greater than $T + 1$ that are first-order optimal, those that maximize the mutual information up to second order, or equivalently, the wideband slope, are characterized.

In both PPAPR- and peak-constrained problems, the optimal solutions are obtained in closed form to finite-dimensional nonconvex optimizations. Moreover, the solutions are established to be both necessary and sufficient to optimize their respective information-theoretic metrics. Interestingly, the solutions to both the PPAPR- and peak-constrained problems

are found to be identical. Due to its special structure, we refer to the common solution as space-time orthogonal rank one modulation, or STORM.

Moreover, in the PPAPR constrained case, STORM (with cardinality $T + 1$) is shown to be near-optimal even among constellations of unconstrained cardinality, even for modest values of T and PAPR. Hence, there is not much to be gained by using more than $T + 1$ points in this case. In the peak-constrained case, we first obtain necessary and sufficient conditions for a constellation of *any* finite cardinality to achieve the minimum energy per bit. Among all such constellations, when the cardinality is no greater than $T + 1$, STORM is established as being both first- and second-order optimal. Our approach provides a far more detailed characterization of the first- and second-order behavior of noncoherent MIMO capacity than in existing literature. Specifically, we show that when the peak power is less than a certain threshold, it is possible to have a wideband slope that is nonzero, and obtain the maximum wideband slope achievable by a $T + 1$ point probability mass function (pmf). Moreover, the energy per bit and the wideband slope achieved by STORM reveal a fundamental energy-versus-bandwidth efficiency tradeoff that enable the determination of the operating (low) SNR and peak power most suitable for a given application.

It also follows from our analysis and optimization that while the conventional MIMO on-off keying (OOK) also achieves the minimum energy per bit, STORM has a wideband slope that is T times greater which translates into an increase in bandwidth efficiency (or a decrease in the PAPR) by a factor of T in the wideband regime. Given typical values of the coherence block length T , these gains are potentially huge. Our results and conclusions also temper the conclusions of [6] obtained under only the average power constraint regarding noncoherent communications over fading channels.

Among the several new insights that STORM provides on communications in the low-SNR regime one that runs contrary to conventional wisdom is that, under the practical constraints considered in this work, it helps to use *all* available transmit antennas, not just one, to transmit linearly dependent signals across them in the low-SNR regime.

Note that in this work, the input distribution is not *a priori* assumed or restricted as it is in most prior work. STORM is obtained through novel techniques involving nonconvex optimization of information-theoretic measures. Consequently, our approach provides necessary and sufficient conditions for a constellation to be optimal for the noncoherent MIMO Rayleigh-fading channel, resolving a long-standing open problem. Low duty cycled M -ary FSK (MFSK) [2], [4], [5] which is often proposed to achieve first-order optimality in a SISO channel, is seen to be closely related to a special case of STORM. However, the zero symbol in STORM is information bearing which is not the case in low duty cycled MFSK. This can make STORM have higher achievable rates especially in the PPAPR-constrained case. Moreover, in this work, we specify a class of STORM constellations. One subtle insight afforded through different STORM constellations is that optimal signal constellations need not be peaky in frequency dimension (as in low duty cycled MFSK), in addition to being peaky in time dimension. In the process, we discover a new optimal SISO

constellation which may be called “permuted MFSK” due to its relation to MFSK but would have better spectral properties in general.

To close this section, some notational conventions used throughout the paper are described. Matrices are denoted by the boldfaced capital letters, and vectors by bold faced lower case letters. The symbol \otimes denotes the Kronecker product. The matrices \mathbf{X}^T , $\bar{\mathbf{X}}$, and \mathbf{X}^* denote the transpose, complex-conjugate, and conjugate transpose of \mathbf{X} , respectively. Moreover, $\text{tr}(\mathbf{X})$ and $|\mathbf{X}|$ denote the trace and determinant of the matrix \mathbf{X} . The notation $[\mathbf{X}]_{ij}$ refers to the (i, j) th element of the matrix \mathbf{X} . The notation $\mathbf{X}^{(m)}$ refers to the m th row of the matrix \mathbf{X} . For an integer N , \mathbf{I}_N is an $N \times N$ identity matrix and $\mathbf{1}_N$ is the N -length column vector of ones. The block diagonal matrix with matrices $\mathbf{A}_1, \dots, \mathbf{A}_N$ along the block diagonal and zeros elsewhere is denoted as $\text{blockdiag}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_N)$. $\mathbb{E}[\cdot]$ denotes the expectation operator. A function $f(\rho)$ is said to behave as $o(\rho)$ when $\lim_{\rho \rightarrow 0} f(\rho)/\rho = 0$. The symbol X^C is used to denote the complement of the set X . The symbol \preceq is used to denote generalized inequality, i.e., if $\mathbf{A} \preceq \mathbf{B}$ then $\mathbf{B} - \mathbf{A}$ is positive semidefinite (psd). The first and second derivatives of a function $f(x)$ at $x = c$ are denoted by $\dot{f}(c)$ and $\ddot{f}(c)$, respectively. The function $\log(\cdot)$ always refers to natural logarithm, unless otherwise specified. Complex, circularly symmetric, Gaussian random vectors with mean \mathbf{m} and covariance matrix \mathbf{Q} are said to be $\mathcal{CN}(\mathbf{m}, \mathbf{Q})$ distributed.

II. SYSTEM MODEL

Consider a MIMO channel with N_t transmit and N_r receive antennas. The random channel matrix $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$ is assumed to be constant for a duration of T symbols after which it changes to an independent value. It is assumed to have i.i.d. $\mathcal{CN}(0, 1)$ entries. The knowledge of the distribution of \mathbf{H} is known to the transmitter and receiver. The realizations of \mathbf{H} however, are unknown at both ends. With the transmitted symbol denoted as $\mathbf{X} \in \mathbb{C}^{T \times N_t}$, the output of the channel can be written as

$$\mathbf{Y} = \mathbf{X}\mathbf{H} + \mathbf{N}. \quad (1)$$

The entries of the additive noise matrix \mathbf{N} are assumed to be i.i.d. $\mathcal{CN}(0, 1)$ distributed random variables. The symbol \mathbf{X} is drawn from a finite constellation or alphabet \mathcal{C} with matrix-valued elements.

Two key cases based on the types of constraints imposed are considered in this work.

1) *PPAPR-Constrained Case*: It is assumed that the average SNR at each receive antenna is constrained to be P so that

$$\frac{1}{T} \mathbb{E}[\text{tr}(\mathbf{X}\mathbf{X}^*)] \leq P. \quad (2)$$

Moreover, a peak-power constraint is imposed per space-time slot, namely

$$\|\mathbf{X}\|_\infty \triangleq \max_{i,j} |[\mathbf{X}]_{ij}| \leq \sqrt{K}, \quad \forall \mathbf{X} \in \mathcal{C}. \quad (3)$$

This is most natural and is a practically meaningful peak-power constraint as it restricts the peak power per antenna and per time

slot (to be at most K). It accurately models constraints on individual transmit radio-frequency (RF) power amplifiers in practice. The PPAPR constraint is that the ratio $\frac{K}{P}$ is taken to be a fixed constant. This condition ensures that as the average SNR $P \rightarrow 0$, the maximum peak power also goes to zero.

2) *Peak-Constrained Case*: Here, the average-power constraint (2) and the peak-power constraint (3) are assumed to hold. In this case, however, K is assumed to be a fixed constant independent of P . In other words, in contrast to the PPAPR-constrained case, the peak power remains constrained by K (and does not change) as the average SNR $P \rightarrow 0$.

For convenience, we will denote the average energy per block of T symbols as $E = PT$.

The noncoherent MIMO Rayleigh-fading channel thus described is completely specified by the input constraints and the transition probability density function (pdf) of \mathbf{Y} conditioned on \mathbf{X} being transmitted and is easily seen to be

$$p(\mathbf{Y}|\mathbf{X}) = \frac{\exp\left\{-\text{tr}\left(\mathbf{Y}^*(\mathbf{I}_T + \mathbf{X}\mathbf{X}^*)^{-1}\mathbf{Y}\right)\right\}}{\pi^{TN_r} |\mathbf{I}_T + \mathbf{X}\mathbf{X}^*|^{N_r}}.$$

Finally, there will also be occasion to use the notion of the PAPR of a constellation \mathcal{C} which is defined as

$$\max_{m,n} \max_{\mathbf{X} \in \mathcal{C}} \frac{|[\mathbf{X}]_{mn}|^2}{\mathbb{E}\left\{|[\mathbf{X}]_{mn}|^2\right\}}. \quad (4)$$

III. MAXIMIZING THE MUTUAL INFORMATION AT LOW SNR UNDER THE PPAPR CONSTRAINT

Consider the above-defined finite-input and continuous-output noncoherent MIMO Rayleigh-fading channel over which the input constellation $\{\mathbf{X}_i\}_{i=1}^L$ is used with corresponding transmission probabilities $\{P_i\}_{i=1}^L$. The mutual information between the transmitted and received signals, normalized by the block length T (in units of nats/dimension), is thus given as

$$I(\mathbf{X}; \mathbf{Y}) = \frac{1}{T} \sum_i P_i \int_{\mathbf{Y}} p(\mathbf{Y}|\mathbf{X}_i) \log\left(\frac{p(\mathbf{Y}|\mathbf{X}_i)}{p(\mathbf{Y})}\right) d\mathbf{Y}. \quad (5)$$

A closed-form expression for $I(\mathbf{X}; \mathbf{Y})$ is unfortunately not known for general SNR. At asymptotically low SNR, however, and when the input signal satisfies certain regularity conditions to avoid inputs being prohibitively peaky, the authors in [9] show that the mutual information is zero up to first order for the continuous input and continuous output counterpart of the above channel. Moreover, the mutual information up to the second order in P is also obtained in closed form through a Taylor series expansion and without any assumption on the signal structure beyond the regularity conditions. Note that the expression for mutual information up to second order was also derived earlier in [10] and [11], but with more stringent conditions on the input distribution.

For the sake of completeness, the key theorem in [9] for the continuous input and continuous output channel, slightly modified to account for the different power normalizations in this paper, is stated next.

Theorem 1 [9, Theorem 1]: Let $p(\mathbf{Y})$ denote the pdf of \mathbf{Y} .

1) *First-Order Result:* If i) $\frac{\partial p(\mathbf{Y})}{\partial P}$ exists at $P = 0$, and ii) $\lim_{P \rightarrow 0} \frac{\mathbb{E}[\text{tr}\{(\mathbf{X}\mathbf{X}^*)^2\}]}{P} = 0$, the mutual information between the transmitted and received signals \mathbf{X} and \mathbf{Y} is zero to first order in P , i.e., $I(\mathbf{X}; \mathbf{Y}) = o(P)$.

2) *Second-Order Result:* If, in addition, i) $\partial^2 p(\mathbf{Y})/\partial P^2$ exists at $P = 0$, ii) $\mathbb{E}[\text{tr}\{(\mathbf{X}\mathbf{X}^*)^2\}] < \infty$, and iii) $\lim_{P \rightarrow 0} \frac{\mathbb{E}[\text{tr}\{(\mathbf{X}\mathbf{X}^*)^3\}]}{P^2} = 0$, then the mutual information between \mathbf{X} and \mathbf{Y} up to second order in P is given by

$$I(\mathbf{X}; \mathbf{Y}) = \frac{N_r}{2T} \text{tr} \left\{ \mathbb{E}[(\mathbf{X}\mathbf{X}^*)^2] - (\mathbb{E}[\mathbf{X}\mathbf{X}^*])^2 \right\} + o(P^2). \quad (6)$$

The applicability of the above result to the discrete input channel with the PPAPR constraint is next discussed. First, following the proof of the above theorem in [9], it can be seen to hold for the discrete input (and continuous output) case and yield the same expression as in (6) for mutual information with the expectations in (6) now over the discrete instead of continuous input as in [9]. The existence of the first and second derivatives of $p(\mathbf{Y})$ at $P = 0$ are easily verified for the problem at hand. With the PPAPR constraint in effect, the peakiness conditions, namely conditions 1)ii and 2)iii and 2)iii of Theorem 1, are also easily verified to hold as well. Hence, it can be concluded that for a discrete input satisfying the PPAPR constraint i) the mutual information is zero up to first order in P and ii) denoting the coefficient of P^2 in the mutual information $I(\mathbf{X}; \mathbf{Y})$ of (5) as I_{low}

$$\begin{aligned} I_{\text{low}} &\triangleq \lim_{P \rightarrow 0} \frac{I(\mathbf{X}; \mathbf{Y})}{P^2} \\ &= \lim_{P \rightarrow 0} \frac{N_r}{2P^2 T} \text{tr} \left\{ \mathbb{E}[(\mathbf{X}\mathbf{X}^*)^2] - (\mathbb{E}[\mathbf{X}\mathbf{X}^*])^2 \right\}. \end{aligned} \quad (7)$$

Evidently, the dominant second-order term in the mutual information at low SNR is $I_{\text{low}}P^2$. The problem of interest is hence to maximize I_{low} over $\{\mathbf{X}_i\}_{i=1}^L$ and $\{P_i\}_{i=1}^L$ under an average-power constraint $\sum_i P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^*) \leq E$ and a peak-power constraint $\|\mathbf{X}_i\|_\infty = \max_{m,n} |\mathbf{X}_i|_{mn} \leq \sqrt{K} \forall i$.

Before unveiling the solution to the above problem, we note that in [9] the mutual information up to second order is maximized over continuous input distributions under two different peak-power constraints. The solutions however rely on the assumption that the input signal has the form

$$\mathbf{S} = \Phi \bar{\mathbf{V}} \quad (8)$$

where Φ is an isotropically distributed unitary random matrix and $\bar{\mathbf{V}}$ is a diagonal (random) matrix with nonnegative entries. While this imposition entails no loss of optimality for the case when only the average power is constrained (which is a seminal result of [18]), it does result in a loss of optimality, and a significant one at that, when the peak-power constraint of [9] is enforced which is that the diagonal entries $|\bar{V}_i|^2 \leq K$. Due to the suboptimal restriction in (8), the maximizations in [9] lead to the

misleading conclusion that it is optimal to use a single transmit antenna in the low-SNR regime. In [11] also, the authors perform the same maximization over continuous input distributions but under a more relaxed peak constraint $\text{tr}(\mathbf{X}\mathbf{X}^*) \leq \epsilon$ and conclude that a single antenna should be used. Different from [9] and [11], the optimization problem considered here does not suboptimally restrict the signals to be as in (8) while considering averaged power constrained discrete inputs and the practically relevant peak-power constraint per space-time slot. These assumptions result in a significantly different and more challenging problem than those considered in [9] or [11]. Indeed, in contrast to [9] or [11], our results indicate that in the PPAPR-constrained problem, at sufficiently low SNR, it actually helps to use *all* transmit antennas.

For the PPAPR-constrained problem, the set of all feasible constellations with cardinality L is denoted as \mathcal{S}_L and can be described as shown in the equation at the bottom of the page. It is assumed, without loss of generality, that $KN_t T \geq E$, because otherwise, the average-power constraint cannot be active and one can therefore solve the problem by changing the average-power constraint to $E' = KN_t T$. Let the PPAPR be denoted as $\zeta = \frac{K}{P}$, a constant in the PPAPR-constrained case as P varies.

Let $I_{\text{low},L}^*$ be the maximum mutual information up to second order achievable by any constellation in the set \mathcal{S}_L , so that

$$I_{\text{low},L}^* = \max_{(\mathbf{X}_i, P_i)_{i=1}^L \in \mathcal{S}_L} I_{\text{low}}. \quad (9)$$

Note that when $P_i = 0$, the symbol \mathbf{X}_i is not used and therefore the set of feasible constellations in $\mathcal{S}_{L'}$ is included in the set \mathcal{S}_L for any $L' < L$. Hence, $I_{\text{low},L}^*$ is the maximum mutual information up to second order achievable by any constellation of cardinality no greater than L . The maximum mutual information up to second order when there is no upper limit on the cardinality of the discrete input constellation is defined as $I_{\text{low}}^* = \lim_{L \rightarrow \infty} I_{\text{low},L}^*$. It will be shown in what is to follow that $I_{\text{low},L}^* \Big|_{L=T+1}$ (and its associated constellation of size $T+1$) is near-optimal in that it can be very close to I_{low}^* (and the as yet unknown constellation which achieves the latter).

The following theorem is one of the main results in this paper.

Theorem 2 (PPAPR-Constrained Case): Let the coherence time $T \geq 2$. When $L \leq T+1$, the maximum second-order mutual information with an L -point input constellation is given as

$$I_{\text{low},L}^* = \frac{N_t N_r T}{2} \left(\zeta - \frac{1}{(L-1)N_t} \right). \quad (10)$$

An L -point constellation (or pmf) achieves $I_{\text{low},L}^*$ with $L \leq T+1$ if and only if (iff) it is of the following form:

$$(\mathbf{X}_i, P_i) = \left(\sqrt{K} \mathbf{v}_i \mathbf{w}_i^*, \frac{E}{(L-1)KN_t T} \right), \quad 1 \leq i \leq L-1 \quad (11)$$

$$\mathcal{S}_L = \left\{ (\mathbf{X}_i, P_i)_{i=1}^L : P_i \geq 0, \mathbf{X}_i \in \mathbb{C}^{T \times N_t}, \sum_{i=1}^L P_i = 1, \sum_{i=1}^L P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^*) \leq E, \|\mathbf{X}_i\|_\infty \leq \sqrt{K}, \forall i \right\}.$$

$$(\mathbf{X}_L, P_L) = \left(\mathbf{0}_{T \times N_t}, 1 - \frac{E}{KN_tT} \right) \quad (12)$$

where for each i , $\mathbf{v}_i \in \mathbb{C}^{T \times 1}$ is the i th column of a unitary matrix \mathbf{V} , $\mathbf{w}_i \in \mathbb{C}^{N_t \times 1}$, and

$$|[\mathbf{v}_i \mathbf{w}_i^*]_{mn}| = 1, \quad \forall i, m, n. \quad (13)$$

Furthermore, I_{low}^* , the maximum second-order mutual information with an unconstrained cardinality, is bounded above and below as

$$I_{\text{low},L}^* \Big|_{L=T+1} = \frac{N_r}{2} (\zeta N_t T - 1) \leq I_{\text{low}}^* < \frac{N_r}{2} \zeta N_t T. \quad (14)$$

Proof: The proof is given in Section III-B. \square

The optimal signal constellation for $L = T + 1$ given in Theorem 2 can be viewed as a space–time code (employing unequal transmission probabilities) that achieves the maximum mutual information up to second order at low SNR. Based on its structure, it is referred to as **Space–Time Orthogonal Rank one Modulation (STORM)** because each nonzero matrix is of unit rank and is orthogonal to the other constellation matrices by construction. Two examples of matrices that can be used for the unitary matrix \mathbf{V} are the discrete Fourier transform (DFT) matrix and the Hadamard matrix (when it exists). In one embodiment of STORM, \mathbf{w}_i can be chosen to be $\mathbf{1}_{N_t} \forall i$. In this case, each of the $L - 1 = T$ nonzero constellation points is formed from a column of \mathbf{V} and this column is repeated over the N_t antennas. The L th point is, of course, the all-zero matrix.

It can be seen that for STORM, the PAPR as defined in (4), is $\frac{KN_t}{P} = \zeta N_t \geq 1$. Clearly, the ratio between the upper and lower bounds on I_{low}^* in (14) is nearly equal to unity when $\zeta N_t T \gg 1$. This is evidently true even for moderate and practical values of PAPR and T . As an example, for $\zeta = 2$, $N_t = 2$, and $T = 4$, the ratio is 0.94. Hence, even for moderate values of PAPR and T , the $T + 1$ point STORM almost achieves I_{low}^* (the limit with unconstrained cardinality) and there is not much to be gained by using more than $T + 1$ points.

A. Remarks

Since STORM achieves a significant fraction of I_{low}^* even for moderate values of η and T , the following insights from its structure and mutual information up to second order it achieves at low SNR are of interest. For brevity, the mutual information up to second order at low SNR is simply referred to as mutual information in the rest of this section.

1. It can be seen that the mutual information of STORM increases linearly with the maximum peak power K . That it is an increasing function is to be expected since peaky signaling is known to achieve the noncoherent capacity in the low-SNR regime when there is only an average power constraint. Moreover, the mutual information also increases linearly as a *product* $N_t \cdot N_r$ of the numbers of transmit and receive antennas. The use of a single antenna is evidently suboptimal by a factor of N_t .

2. A reason that is often cited in the literature for explaining the efficacy of using a single antenna at low SNR is that the number of channel parameters that are to be implicitly estimated

is the least in this case. The use of a single antenna however is not necessary to ensure this and can even be detrimental to performance as explained above. Consider STORM, where the received signal when the i th nonzero signal is transmitted is

$$\mathbf{Y} = \sqrt{K} \mathbf{v}_i \mathbf{w}_i^* \mathbf{H} + \mathbf{N} = \sqrt{K} \mathbf{v}_i \mathbf{h}^T + \mathbf{N} \quad (15)$$

where $\mathbf{h}^T = \mathbf{w}_i^* \mathbf{H}$ and so \mathbf{h} is $\mathcal{CN}(\mathbf{0}, N_t \mathbf{I}_{N_r})$ distributed. Therefore, the *effective* channel (15) does in fact involve only N_r (and not $N_t N_r$) unknown channel coefficients even though all transmit antennas are used. The optimality of the unit rank structure of STORM could thus be indeed attributed to the difficulty of (implicit) estimation of $N_t N_r$ coefficients at low SNR because it avoids this task by focusing the power on just N_r effective unknown path gains, while at the same time making use of all the transmit antennas.

3. Consider the case when $\mathbf{w}_i = \mathbf{1}_{N_t} \forall i$ in (11), which is sufficient for $T + 1$ point STORM to be optimal. Then the symbols sent by all transmit antennas at any given time are identical and the fading gains effectively add up at each receive antenna. So, why not just use a single transmit antenna? All transmit antennas must be used because otherwise the effective received power is smaller due to the peak-power constraint which limits the symbol power per antenna and per time slot.

4. A canonical embodiment of STORM is one that results from setting $\mathbf{w}_i = \mathbf{1}_{N_t} \forall i$ and $\mathbf{V} = [\mathbf{v}_1 \cdots \mathbf{v}_T]$ to be a T -dimensional DFT matrix in (11). A convenient feature in this DFT version of STORM is that the entries of the signal matrices can be transmitted using PSK symbols with an additional zero point. Alternatively, a T -dimensional Hadamard matrix can be used for \mathbf{V} (when it exists). The advantage of using a Hadamard matrix is that it is enough to transmit real symbols for *each* entry, specifically, binary phase-shift keying (BPSK) and an additional zero point. Hadamard matrices of dimension T exist when $T = 2^n$ for any natural number n and also for many multiples of 4. In Appendix B, we show how block decoding of STORM may be simplified using either the fast Fourier transform (FFT) or the fast Hadamard transform (FHT), when $L - 1$ is a power of 2.

5. Consider the special case when there is only a peak constraint on the input (i.e., $KN_tT = E$). Here, it can be seen that STORM has no zero point (so $L = T$) and is given by

$$(\mathbf{X}_i, P_i) = \left(\sqrt{K} \mathbf{v}_i \mathbf{w}_i^*, \frac{1}{L} \right), \quad 1 \leq i \leq L. \quad (16)$$

Hence, all points are equiprobable and the PAPR is unity, thus facilitating practical implementation. Moreover, this constellation is near-optimal when there is only a peak constraint and when $T \gg 1$ as seen from the bounds on I_{low}^* in (14) of Theorem 2.

6. The canonical version of STORM can be seen as a form of generalized $(T + 1)$ -ary ON–OFF signaling with repetition coding across the transmit antennas and with unequal probabilities of ON and OFF signaling, with the ON signaling actually being the classical T -ary, equiprobable frequency-shift keying (TFSK). The larger the allowed PPAPR, the higher the probability of the OFF signal. In fact, STORM takes advantage of all

the peak power allowed for each space–time slot when transmitting nonzero symbols while meeting the average-power constraint by the inclusion of the zero symbol with as high a probability as the PPAPR constraint would allow.

7. Consider the special case of a SISO system when there is only a per time-slot peak-power constraint K . Here, Theorem 2 establishes the second-order optimality of equiprobable TFSK at low SNR among all T -ary constellations, and the near-optimality under unconstrained cardinality when $T \gg 1$. In general, for SISO systems however, depending on the peak-power and average-power constraints, an additional zero signal is needed of probability different from that of the T (equiprobable) TFSK signals.

8. The mutual information of STORM may be expressed as $\frac{EN_r}{2}(KN_t - \frac{E}{T^2})$. For fixed K , T , and E , it increases linearly with N_t . This may be attributed to the fact that increasing N_t with fixed K , T , and E increases the overall peak power $\text{tr}(\mathbf{X}\mathbf{X}^*) = \zeta N_t E$, while simultaneously decreasing the probability of transmitting a nonzero signal $\frac{1}{\zeta N_t}$, thereby making the signals more peaky in the time domain. On the other hand, when T is increased for a fixed N_t and E , the overall peak-power $\text{tr}(\mathbf{X}\mathbf{X}^*) = \zeta N_t E$ and the probability of transmitting the zero signal $\frac{1}{\zeta N_t}$ remain fixed but the mutual information increases with T . To get some insight on why this is so, consider the canonical version of STORM. An increase in T implies that the TFSK transmissions (repeated over each antenna) become more peaky in the frequency domain.¹

9. STORM constellations other than the canonical ones can also be constructed. For example, one can use the inverses of the DFT and Hadamard matrices for a choice of \mathbf{V} . More generally, if $\tilde{\mathbf{V}}$ is unitary with unit-magnitude elements so is $\mathbf{V} = \mathbf{P}\tilde{\mathbf{V}}\mathbf{Q}$ where \mathbf{P} and \mathbf{Q} are $T \times T$ permutation matrices. \mathbf{Q} only permutes the columns of $\tilde{\mathbf{V}}$ thereby renumbering the signals leaving the STORM constellation unchanged. However, row permutations induced by \mathbf{P} would result in constellations that are no longer peaky in the frequency domain as compared to the canonical DFT version of STORM. It is unclear as to how the complete class of STORM constellations can be constructed. In this regard, note that the \mathbf{w}_i vectors can be arbitrary as long as its elements have unit magnitudes. So “repetition” across transmit antennas can involve arbitrary phase rotations or multiplication by possibly distinct unit-magnitude complex numbers.

10. The cutoff rate for the discrete input (of cardinality L) and continuous output channel is given by

$$R_0 = \max_{\{P_i\}_{i=1}^L, \{\mathbf{X}_i\}_{i=1}^L} -\log \left\{ \sum_i \sum_j P_i P_j \int \sqrt{p(\mathbf{y}|i)p(\mathbf{y}|j)} d\mathbf{y} \right\}. \quad (17)$$

The cutoff rate was initially advocated as a design criterion for modulation schemes in [19] and [20]. It is a lower bound on the random coding exponent, and also provides an exponentially accurate description of the attainable error probability when communicating at the critical rate [19]. Let the argument of $\max(\cdot)$

¹This was pointed out by a reviewer.

in (17) be denoted as the *cutoff rate expression*. For the noncoherent MIMO channel at low SNR, the cutoff rate expression is easily shown to be (cf. [21])

$$CR_{\text{low}} = \frac{N_r}{8} \sum_{i,j} P_i P_j \text{tr} \left\{ (\mathbf{X}_i \mathbf{X}_i^* - \mathbf{X}_j \mathbf{X}_j^*)^2 \right\} + o(P^2). \quad (18)$$

An interesting property of CR_{low} [16] is that when the input constellation satisfies the regularity conditions,

$$\lim_{P \rightarrow 0} CR_{\text{low}}/I_{\text{low}} = \frac{1}{2}.$$

In the limit of low SNR, therefore, CR behaves identically to the mutual information. Therefore, the $T+1$ -point STORM also maximizes the cutoff rate expression up to second order at low SNR.

11. An often used noncoherent constellation design criterion (cf. [22], [23]) is to maximize the worst case chordal distance which is given by $\min_{j \neq i} \text{tr} \{ \mathbf{I} - \mathbf{X}_i^* \mathbf{X}_j \mathbf{X}_j^* \mathbf{X}_i \}$. For STORM, the worst case chordal distance is the maximum possible as for every $i \neq j$, $\mathbf{X}_i^* \mathbf{X}_j = \mathbf{0}_{N_t \times N_t}$. Moreover, the difference between any two distinct matrices in STORM has unit rank, and hence the scheme would have a diversity order of N_r at high SNR if employed as a coherent space–time code [24] whereas constellation design at high SNR for the coherent MIMO channel is typically geared towards achieving maximum diversity ($N_t N_r$). Theorem 2 shows that optimal noncoherent constellations at low SNR have quite the opposite properties from good coherent constellations at high SNR.

12. Subsequent to the conference version of this paper [25] (see also [26]), Sethuraman *et al.* [27] consider a MIMO Rayleigh-fading channel with the noncoherent assumption and with the fading process modeled as stationary and ergodic, as well as correlated over time. The authors characterize input distributions which are optimal for the stationary and ergodic MIMO channel, under average-power constraints and peak-constraints which are per space–time slot similar to the PPAPR-constrained case here. Interestingly, one distribution identified in [27] which achieves the capacity up to second order can be seen to be closely related to the canonical version of STORM here. While this distribution is obtained for a different fading process, the channel coherence time T here can be thought of as playing the same role as channel memory in [27].

B. Proof of Theorem 2

In this subsection, the proof of Theorem 2 is given. The following definitions and lemmas are needed first from [28].

Definition 1: A convex maximization problem is an optimization problem in the following form:

$$\max_{\mathbf{x} \in X} f(\mathbf{x}) \quad (19)$$

where $f(\mathbf{x})$ is a convex function and $X \subset \mathfrak{R}^n$ is a convex set.

Definition 2: A point \mathbf{x} on the boundary of a convex set X is called an extreme point if there are no distinct points $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2$, $0 < \lambda < 1$.

Lemma 1: A closed, bounded convex set in \mathbb{R}^n is the convex hull of its extreme points.

Lemma 2: The global maximum of a convex function f over a compact convex set X is attained at an extreme point of X . A point in a compact convex set X is a global maximizer of a strictly convex function f if it is an extreme point of X .

Definition 3: A *polyhedron* is defined to be the set of points $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \text{ where } \mathbf{A} \in \mathbb{R}^{m \times n} \text{ and } \mathbf{b} \in \mathbb{R}^m\}$. A bounded polyhedron is called a *polytope*. The extreme points of a polytope are referred to as *vertices*.

The next lemma gives the necessary and sufficient conditions for a point to be a vertex of a general polytope.

Lemma 3: With the same notation as in Definition 3, let $\mathbf{a}_i^T, 1 \leq i \leq m$ denote the rows of the matrix \mathbf{A} . Further, for $\mathbf{x} \in \mathcal{P}$, let $I = \{i \in \{1, \dots, m\} : \mathbf{a}_i^T \mathbf{x} = b_i\}$ describe the inequalities which are binding (active) at \mathbf{x} , and let \mathbf{A}_I be the matrix with rows $\mathbf{a}_i^T, i \in I$. Then $\mathbf{x} \in \mathcal{P}$ is a vertex of \mathcal{P} iff $\text{rank}(\mathbf{A}_I) = n$.

The following lemma more sharply specifies the vertices of a special polytope which will be useful in the proof of Theorem 2.

Lemma 4: Consider the polytope defined by

$$\mathcal{D} = \left\{ \mathbf{d} : \sum_i P_i d_i \leq E, 0 \leq d_i \leq Q, i = 1, \dots, L \right\} \quad (20)$$

which is the intersection of the half-plane $\sum_i P_i d_i \leq E$ and the hypercube $0 \leq d_i \leq Q$. Each vertex of \mathcal{D} consists of $L - 1$ entries that are either Q or 0 , and exactly one entry c such that $0 \leq c \leq Q$.

Proof: The polytope \mathcal{D} can be expressed in the standard form $\mathbf{A}\mathbf{d} \leq \mathbf{b}$ given in Definition 3, by setting

$$\mathbf{A} = \begin{bmatrix} \mathbf{q}^T \\ \mathbf{I}_L \\ -\mathbf{I}_L \end{bmatrix}_{(2L+1) \times L} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} E \\ Q \mathbf{1}_L \\ \mathbf{0}_L \end{bmatrix} \quad (21)$$

where $\mathbf{q} = [P_1 \ P_2 \ \dots \ P_L]^T$. Let \mathbf{x} be a vertex of the polytope described by $\mathbf{A}\mathbf{d} \leq \mathbf{b}$. Then, the rows of \mathbf{A} which satisfy $\mathbf{a}_i^T \mathbf{x} = b_i$ should form a matrix with rank L by Lemma 3. If \mathbf{x} is a vertex for which $\mathbf{q}^T \mathbf{x} = E$ then there are at least $L - 1$ more linearly independent rows of \mathbf{A} that correspond to active constraints. Suppose k of them are of the form $x_j = Q$ for $j \in J \subseteq \{1, 2, \dots, L\}$, then at least $L - 1 - k$ active constraints (out of the remaining $L - k$ constraints) must be of the form $x_j = 0$ for $j \in J^C$. Hence, at most one entry of \mathbf{x} can lie anywhere between 0 and Q (call it c). If \mathbf{x} is such that $\mathbf{q}^T \mathbf{x} < E$, then of course it is a vertex by Lemma 3 iff $x_j = Q$ for all j in the subset $J \subseteq \{1, 2, \dots, L\}$ for which $Q \sum_{j \in J} P_j < E$ and $x_j = 0$ for all $j \in J^C$ (there are as many such vertices as there are subsets J for which $Q \sum_{j \in J} P_j < E$). In this case, all the entries of the vertex are either Q or 0 (set $c = 0$ or Q). \square

Proof (of Theorem 2): The problem that needs to be solved here is essentially

$$\max_{\{P_i\}_{i=1}^L, \{\mathbf{X}_i\}_{i=1}^L} I_{\text{low}}$$

$$\begin{aligned} \text{subject to } & \sum_i P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^*) \leq E \\ & \|\mathbf{X}_i\|_\infty \leq \sqrt{K}, \forall i \\ & \sum_i P_i = 1, P_i \geq 0 \forall i \end{aligned} \quad (22)$$

where I_{low} is given in (7). Maximizing I_{low} is equivalent to maximizing

$$\sum_i P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_i \mathbf{X}_i^*) - \text{tr} \left(\sum_i P_i \mathbf{X}_i \mathbf{X}_i^* \sum_j P_j \mathbf{X}_j \mathbf{X}_j^* \right) \quad (23)$$

$$\begin{aligned} &= \sum_i P_i (1 - P_i) \text{tr}(\mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_i \mathbf{X}_i^*) \\ &\quad - \sum_{i, j \neq i} P_i P_j \text{tr}(\mathbf{X}_j^* \mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_j) \end{aligned} \quad (24)$$

$$\leq \sum_{i=1}^L P_i (1 - P_i) \text{tr}(\mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_i \mathbf{X}_i^*) . \quad (25)$$

Since terms of the form $\text{tr}(\mathbf{X}_j^* \mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_j)$ are nonnegative, (25) follows by replacing all negative terms in (24) by zero. Let \mathbf{x}_{ik} denote the k th column of the matrix \mathbf{X}_i . The equality in (25) occurs iff $\mathbf{x}_{jk}^* \mathbf{x}_{il} = 0 \forall k, l, j \neq i$. The strategy is to maximize the bound in (25) and show later that the signal constellation that maximizes it achieves equality in the inequality in (25) when $L \leq T + 1$, thereby maximizing I_{low} in these cases. So, let us consider the optimization problem

$$\begin{aligned} & \max_{\{P_i\}_{i=1}^L, \{\mathbf{X}_i\}_{i=1}^L} \sum_i P_i (1 - P_i) \text{tr}(\mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_i \mathbf{X}_i^*) \\ & \text{subject to } \sum_i P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^*) \leq E \\ & \quad \|\mathbf{X}_i\|_\infty \leq \sqrt{K}, \forall i \\ & \quad \sum_i P_i = 1, P_i \geq 0 \forall i. \end{aligned} \quad (26)$$

In Appendix A, a simple argument is given that shows that the maximization of (26) is a nonconvex optimization problem. A two-stage approach is thus adopted for solving the optimization in (26). In the first stage, the objective function is maximized over $\{\mathbf{X}_i\}_{i=1}^L$ while holding $\{P_i\}_{i=1}^L$ fixed. In the second stage, the resulting objective function is maximized over $\{P_i\}_{i=1}^L$. Furthermore, it is shown that the optimization in the first stage can be split into two successive convex maximization problems and the optimization in the second stage is a convex minimization problem. It is this nice structure that is exploited to obtain the signal matrices $\{\mathbf{X}_i\}_{i=1}^L$ and the probabilities $\{P_i\}_{i=1}^L$ that jointly optimize the upper bound on mutual information (up to second order at low SNR) in (26).

Consider first the optimization in (26) over $\{\mathbf{X}_i\}_{i=1}^L$ for fixed $\{P_i\}_{i=1}^L$. This problem is decomposed into two steps. In the first step, $\text{tr}(\mathbf{X}_i \mathbf{X}_i^*) = d_i$ is fixed for some $\{d_i\}_{i=1}^L$ and the best set of $\{\mathbf{X}_i\}_{i=1}^L$ is found. Note that d_i is equal to the energy of the i th signal and because of the peak-power constraint, it is sufficient to restrict $d_i \in [0, KN_t T]$. In the second step, the resulting objective function is optimized over $d_i, i = 1, \dots, L$. Geometrically, we first find the matrices $\{\mathbf{X}_i\}_{i=1}^L$ that maximize the

objective function over the contour $\text{tr}(\mathbf{X}_i \mathbf{X}_i^*) = d_i \forall i$ and then optimize the resulting objective over $\{d_i\}_{i=1}^L$, thereby obtaining the best contour for an arbitrary but fixed $\{P_i\}_{i=1}^L$. As it is shown below, both these problems can be solved as convex maximization problems.

With $\text{tr}(\mathbf{X}_i \mathbf{X}_i^*) = d_i \in [0, KN_t T]$, $\forall i$, it is clear that the objective function in (26) is maximized when for each i , \mathbf{X}_i is chosen according to

$$\max_{\substack{\text{tr}(\mathbf{X}_i \mathbf{X}_i^*)=d_i \\ \|\mathbf{X}_i\|_\infty \leq \sqrt{K}, \forall i}} \text{tr}(\mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_i \mathbf{X}_i^*) . \quad (27)$$

Let the eigenvalues of the positive semidefinite matrix $\mathbf{X}_i \mathbf{X}_i^*$ be $\{\lambda_m\}_{m=1}^T$ (the dependence on i is implicit). Then, the solution of (27) is upper-bounded by the solution of

$$\max_{\substack{\sum_m \lambda_m = d_i \\ \lambda_m \geq 0, \forall m}} \sum_m \lambda_m^2 \quad (28)$$

with equality iff the additional constraints $\|\mathbf{X}_i\|_\infty \leq \sqrt{K}$ hold for each i for the matrix that achieves the maximum in (28). Since the objective function in (28) is strictly convex while the constraint set is a polytope, the problem in (28) is a strictly convex maximization problem. Hence, by Lemma 2, a solution is globally optimal if it is a vertex of the constraint set. In this case, the constraint polytope has $T + 1$ vertices which can be found by inspection to be

$$\begin{aligned} & [0 \ 0 \ \dots \ 0 \ 0]^T, [d_i \ 0 \ \dots \ 0 \ 0]^T, \\ & [0 \ d_i \ 0 \ \dots \ 0]^T, \dots, [0 \ \dots \ 0 \ 0 \ d_i]^T \end{aligned} \quad (29)$$

since none of them can be expressed as a convex combination of any other points in the set, and any point in the set can be expressed as a convex combination of the points in (29). Now, since all the vertices except the all-zero vector give the same value d_i^2 for the objective function, d_i^2 is the sought maximum. This in turn implies that all the matrices $\{\mathbf{X}_i\}_{i=1}^L$ have to be of *unit rank* for the objective functions to achieve their maximum value of d_i^2 for each i (we adopt the convention that the all-zero matrix is of unit rank). Let the number of matrices in $\{\mathbf{X}_i\}_{i=1}^L$ which are not $\mathbf{0}_{T \times N_t}$ be L' . If more than one of the d_i 's are zero, they would all correspond to the same zero signal point $\mathbf{0}_{T \times N_t}$ and their respective probabilities would simply add up, resulting in one effective zero symbol matrix. Therefore, $L = L' + 1$ or $L = L'$ depending on whether or not there is a zero symbol.

When $L' \leq T$, consider the following constellation $\{\mathbf{X}_i\}_{i=1}^L$:

$$\mathbf{X}_i = \sqrt{\frac{d_i}{TN_t}} \mathbf{v}_i \mathbf{w}_i^*, \quad d_i > 0 \quad (30)$$

$$\mathbf{X}_i = \mathbf{0}, \quad d_i = 0 \quad (31)$$

where the vectors \mathbf{v}_i and \mathbf{w}_i are constrained as in (13). Note that the set of matrices in (30), (31) are of unit rank and satisfy $\text{tr}(\mathbf{X}_i \mathbf{X}_i^*) = d_i$, $1 \leq i \leq T$. Hence, they solve the problem in (28). Now, since $d_i \leq KN_t T$, using (13), it follows that $\|\mathbf{X}_i\|_\infty \leq \sqrt{K} \forall i$ and hence they also solve the problem in (27). Moreover, since $L' \leq T$, any pair of different constellation matrices have orthogonal columns (since \mathbf{v}_i 's are orthogonal), which ensures that (25) holds with equality. It will eventually be

shown that the optimal values of the nonzero $\{d_i\}$ are all equal with $d_i = KN_t T \forall i$. This in turn implies that the structure in (30) and (31) is also necessary.

When $L' > T$, the set of \mathbf{v}_i in (30) can no longer be selected to be orthogonal to each other. Nevertheless, a set of rank-one matrices with the structure given in (30) but with a nonorthogonal set of \mathbf{v}_i (normalized in the same way), still solves both (27) and (28). Therefore, the expression $\frac{N_r}{2T} \sum_{i=1}^L P_i (1 - P_i) d_i^2$ serves as an upper bound on the maximum mutual information up to second order achievable by any constellation of cardinality of $L = L' + 1$, which is $I_{low,L}^*$.

In summary, the best constellation $\{\mathbf{X}_i\}_{i=1}^L$ can be specified for any set of nonnegative $\{d_i\}_{i=1}^L$. It remains to find the best $\{d_i\}_{i=1}^L$ according to

$$\max_{\{d_i\}_{i=1}^L} \sum_i P_i (1 - P_i) d_i^2. \quad (32)$$

$$\text{subject to } \sum_i P_i d_i \leq E \quad (33)$$

$$0 \leq d_i \leq KN_t T \forall i. \quad (34)$$

For a fixed $\{P_i\}_{i=1}^L$, this is also a strictly convex maximization problem over a polytope. Hence, a vertex of the polytope is necessary to achieve the global optimum. The polytope constraint set is exactly of the form considered in Lemma 4 which states that each vertex would consist of $L - 1$ entries that are either $KN_t T$ or 0, and at most one entry c such that $0 < c < KN_t T$. For vertices for which $\sum_i P_i d_i < E$, it is necessarily the case that all entries are either 0 or $KN_t T$.

Consider the second stage of the optimization which is over $\{P_i\}_{i=1}^L$. Following the result of the optimization in the first stage, the structure of the optimal \mathbf{d} and the corresponding probabilities are of the form

$$\mathbf{d} = \left[\underbrace{KN_t T \dots KN_t T}_{M \text{ times}} \quad c \quad 0 \right]^T, \quad 0 < c \leq KN_t T \quad (35)$$

$$\mathbf{P} = [P_1 \ \dots \ P_M \ P_{M+1} \ P_{M+2}]^T \quad (36)$$

where M denotes the number of entries in \mathbf{d} that are equal to $KN_t T$. Note that when $P_{M+1} = 0$, the constellation point corresponding to the entry c such that $0 < c < KN_t T$ is not transmitted. We know that whenever (33) is strict, there cannot be an extreme point \mathbf{d} of the constraint set formed by (33) and (34), which has an entry c such that $0 < c < KN_t T$. Therefore, in the case of a strict half-plane constraint, we will take $P_{M+1} = 0$ for the optimal constellation without any loss of generality, which simplifies the subsequent convex minimization problem. The cardinality of the constellation L depends on the number of nonzero probabilities in the optimal constellation and is related to M by $L \leq M + 2$ in general.

With the structure of the optimal \mathbf{d} , the optimal set of probabilities are determined next in terms of M and c . Following that, the values of M and c are obtained that maximize the resulting objective function. For convenience, consider minimizing the negative of the objective function in (32) after the optimal \mathbf{d} is substituted as follows:

$$\min_{\{P_i\}_{i=1}^{M+2}} -K^2 N_t^2 T^2 \sum_{i=1}^M P_i (1 - P_i) - c^2 P_{M+1} (1 - P_{M+1}). \quad (37)$$

$$\text{subject to } KN_tT \sum_{i=1}^M P_i + cP_{M+1} \leq E \quad (38)$$

$$\sum_{i=1}^{M+2} P_i = 1, P_i \geq 0, 1 \leq i \leq M+2. \quad (39)$$

The optimization over \mathbf{P} in (37) is the more commonly studied convex minimization problem [29]. The Lagrangian can be written as

$$\begin{aligned} L(\mathbf{P}, \beta, \lambda, \{\mu_i\}_{i=1}^{M+2}) &= -K^2 N_t^2 T^2 \sum_{i=1}^M P_i (1 - P_i) \\ &\quad - c^2 P_{M+1} (1 - P_{M+1}) + \beta \left(\sum_{i=1}^{M+2} P_i - 1 \right) \\ &\quad + \lambda \left\{ KN_tT \sum_{i=1}^M P_i + cP_{M+1} - E \right\} - \sum_{i=1}^{M+2} \mu_i P_i. \quad (40) \end{aligned}$$

It can be verified that Slater's conditions [29] are satisfied and hence, strong duality holds. Therefore, the Karush–Kuhn–Tucker (KKT) conditions are both necessary and sufficient for the optimal solution \mathbf{P} and are given as the expressions at the bottom of the page. By eliminating the slack variable μ , we get

$$K^2 N_t^2 T^2 (2P_i - 1) + \lambda KN_tT + \beta \geq 0, 1 \leq i \leq M \quad (41)$$

$$c^2 (2P_{M+1} - 1) + \lambda c + \beta \geq 0 \quad (42)$$

$$\beta \geq 0 \quad (43)$$

$$\lambda \left(KN_tT \sum_{i=1}^M P_i + cP_{M+1} - E \right) = 0 \quad (44)$$

$$\beta P_{M+2} = 0 \quad (45)$$

$$(K^2 N_t^2 T^2 (2P_i - 1) + \lambda KN_tT + \beta) P_i = 0, 1 \leq i \leq M \quad (46)$$

$$(c^2 (2P_{M+1} - 1) + \lambda c + \beta) P_{M+1} = 0 \quad (47)$$

$$P_i \geq 0, 1 \leq i \leq M+2 \quad (48)$$

$$\sum_{i=1}^{M+2} P_i = 1 \quad (49)$$

$$KN_tT \sum_{i=1}^M P_i + cP_{M+1} \leq E. \quad (50)$$

From (46), it can be seen that P_i can take one of two values, namely

$$P_i = 0 \quad \text{or} \quad P_i = \frac{1}{2} - \frac{\lambda KN_tT + \beta}{2K^2 N_t^2 T^2}. \quad (51)$$

Points with zero probability are redundant and since the optimal number M is determined only later, it may be assumed that the M probabilities P_i for $1 \leq i \leq M$ are the same and given in (51) and denote these probabilities simply as “ P_i .” Similarly, from (47), P_{M+1} can take one of two values, namely

$$P_{M+1} = 0 \quad \text{or} \quad P_{M+1} = \frac{1}{2} - \frac{\lambda c + \beta}{2c^2}. \quad (52)$$

Four cases must be considered to find the solutions to the KKT conditions. Recall that $KN_tT \geq E$.

$$\text{Case 1: } KN_tT \sum_{i=1}^M P_i + cP_{M+1} < E, P_{M+2} = 0$$

The strict inequality in (50) implies that $\lambda = 0$ from (44). Since the power constraint is a strict inequality, we may take $P_{M+1} = 0$ from the discussion that follows (36). Therefore, $P_i = 1/M$ is necessary to satisfy (49). From (51), we obtain $\beta = M - 2/MK^2 N_t^2 T^2$. The condition $\beta \geq 0$ implies that $M \geq 2$. The strict inequality in (50) together with $P_{M+1} = 0$ implies that this case holds when $KN_tT < E$, which is never true. Therefore, this case does not occur.

$$\text{Case 2: } KN_tT \sum_{i=1}^M P_i + cP_{M+1} < E, P_{M+2} > 0$$

The strict inequality in (50) implies that $\lambda = 0$ from (44). Since the power constraint is a strict inequality, we may take $P_{M+1} = 0$ from the discussion that follows (36). Since $P_{M+2} > 0$, we have $\beta = 0$ from (45). Therefore, $P_i = 1/2$ from (51) and $P_{M+2} = 1/2$. From (50), this case applies when $KN_tT < 2E$ and $M = 1$.

$$\text{Case 3: } KN_tT \sum_{i=1}^M P_i + cP_{M+1} = E, P_{M+2} > 0$$

Since $P_{M+2} > 0$, we must have $\beta = 0$ by (45). There are three subcases here, viz., i) $P_i > 0, P_{M+1} > 0$, ii) $P_i > 0, P_{M+1} = 0$, and iii) $P_i = 0, P_{M+1} > 0$. We first consider subcase i).

i) Using the values $P_i = \frac{1}{2} - \frac{\lambda}{2KN_tT}$ and $P_{M+1} = \frac{1}{2} - \frac{\lambda}{2c}$ from (51) and (52) in the power constraint equality, we can solve for λ as $\lambda = \frac{MKN_tT + c - 2E}{M+1}$. Substituting this value of λ in (51) and (52), we obtain

$$P_i = \frac{KN_tT - c + 2E}{2(M+1)KN_tT} \quad (53)$$

$$\begin{aligned} \lambda \geq 0, \mu_i \geq 0 \forall i, KN_tT \sum_{i=1}^M P_i + cP_{M+1} &\leq E \\ \sum_{i=1}^{M+2} P_i = 1, \lambda \left\{ KN_tT \sum_{i=1}^M P_i + cP_{M+1} - E \right\} &= 0, \quad \mu_i P_i = 0, 0 \leq i \leq M+2 \\ -K^2 N_t^2 T^2 (1 - 2P_i) + \lambda KN_tT + \beta - \mu_i &= 0, \quad 1 \leq i \leq M \\ -c^2 (1 - 2P_{M+1}) + \lambda c + \beta - \mu_{M+1} &= 0, \\ \beta - \mu_{M+2} &= 0. \end{aligned}$$

$$P_{M+1} = \frac{M(c - KN_tT) + 2E}{2c(M+1)}. \quad (54)$$

Using the above probabilities in the objective function f given in (37), we observe that

$$\frac{d^2 f}{dc^2} = -\frac{M}{2(M+1)} \leq 0 \quad (55)$$

which means that f is a concave function over c . Since $P_{M+1} \geq 0$, we get from (52) that $\lambda \leq c$. Therefore, the range of c in this case is given by $\lambda \leq c \leq KN_tT$. Since the optimization of f over c is a concave minimization problem, the minimum is either at $c = \lambda$ or $c = KN_tT$ by Lemma 2.

Choosing $c = \lambda$ gives $P_{M+1} = 0$ from (52), $\lambda = KN_tT - \frac{2E}{M}$ and therefore

$$P_i = \frac{E}{MKN_tT}. \quad (56)$$

Consequently, from (49) we get that

$$P_{M+2} = 1 - \frac{E}{KN_tT}. \quad (57)$$

If c were instead chosen to be KN_tT , then from (53) and (54), $P_i = \frac{E}{(M+1)KN_tT}$, $P_{M+1} = \frac{E}{(M+1)KN_tT}$, and, therefore, $P_{M+2} = 1 - \frac{E}{KN_tT}$. Since we are yet to optimize over M , the above solution clearly is identical to that obtained in (56) and (57). So we may choose $c = \lambda$ itself as the solution.

For $c = \lambda$, since $\lambda \geq 0$, this case requires $KN_tT \geq \frac{2E}{M}$. Moreover, the power constraint equality requires that $KN_tT \geq E$. Hence, this subcase solves the convex optimization problem for the cases $KN_tT \geq E$, $M \geq 2$, and $KN_tT \geq E$, $M = 1$.

Even for subcases ii) and iii), it can be easily verified that we get essentially the same solutions as the previous subcase.

$$\text{Case 4: } KN_tT \sum_{i=1}^M P_i + cP_{M+1} = E, P_{M+2} = 0$$

The cases $KN_tT \geq E$, $M \geq 2$, and $KN_tT \geq E$, $M = 1$ are solved completely through Cases 2 and 3. This is true because by strong duality, the constellations obtained in Cases 2 and 3 are both necessary and sufficient for optimality. Moreover, since $KN_tT < E$ does not occur, we do not solve for Case 4 since we will get no new solutions or insights.

The last step is to find the best possible M . We revert to the problem which is a maximization of the objective function f for convenience. From Case 3, which yields the only pertinent solution for $T \geq 2$, the objective function with the optimal probabilities given in (56) and (57) is

$$f = KN_tTE \left(1 - \frac{E}{MKN_tT} \right). \quad (58)$$

Notice that f is an increasing function of M , and M needs to be chosen as large as possible. However, if M is chosen so that $M > T$, inequality (25) would be strict since it is not possible to make the columns of all pairs of different constellation matrices orthogonal. Therefore, $M = T$ is optimal among M satisfying $M \leq T$. When we take the limit as $M \rightarrow \infty$, we get an upper bound on the mutual information which is not achievable (hence the strict inequality for the upper bound in (14)).

To complete the proof, notice that we may use the jointly optimal \mathbf{P} and \mathbf{d} with the structure of constellation points given in (30),(31) so that the upper bound in (25) is achieved with

equality when $M \leq T$. Therefore, the optimal constellations have been obtained for the case $M \leq T$. When $M \geq T$, we can obtain an upper bound on the maximum achievable mutual information by letting $M \rightarrow \infty$ in (58) (and multiplying by the factor $N_r/2T$). \square

C. Spectral Efficiency

Consider the normalized energy per bit for reliable communications which is given as

$$\frac{E_b}{N_0} = \frac{P}{C(P)} \quad (59)$$

where $C(P)$ is the Shannon capacity for the channel in bits per dimension. For the case when $C(P)$ is a nondecreasing concave function, it can be seen that (59) achieves its minimum value over all P , as $P \rightarrow 0$. However, this is not true in the PPAPR-constrained case. Indeed, since the capacity is $O(P^2)$, $E_b/N_0 \rightarrow \infty$ as $P \rightarrow 0$. Therefore, it is not energy efficient to operate at asymptotically low SNR in this case. The mutual information of STORM at any SNR is

$$I^{\text{STORM}}(P) = \sum_i P_i E_{\mathbf{Y}|\mathbf{X}_i} \left[\log \left(\frac{p(\mathbf{Y}|\mathbf{X}_i)}{\sum_j P_j p(\mathbf{Y}|\mathbf{X}_j)} \right) \right]. \quad (60)$$

The expectations in (60) can be calculated using Monte Carlo integration. Thus, the normalized energy per bit required for STORM can be determined as $E_b/N_0 = P/I^{\text{STORM}}(P)$, over the entire range of SNRs. It can be seen through extensive simulations over a variety of cases that the minimum energy per bit typically occurs at a low but nonvanishing SNR. STORM should hence be used in the vicinity of this SNR, for maximum spectral efficiency. In the absence of the capacity of the noncoherent MIMO channel at a general SNR, however, there is no fair yardstick to compare the energy per bit of STORM against that of the capacity achieving scheme.

IV. THE PEAK-CONSTRAINED CASE

In this section, the *peak-constrained* problem is considered where the peak constraint K in (3) is a fixed constant, *independent* of the average power P . It can be shown by a simple time-sharing argument that the channel capacity in this case is concave and nondecreasing in P . Therefore, the normalized energy per bit $\frac{E_b}{N_0}$ given in (59) can be seen to attain its minimum value over all P , as $P \rightarrow 0$. Let us denote the normalized minimum energy per bit for our channel model by $\frac{E_b}{N_0 \min}$, in keeping with common usage [6]. Since $C(P)$ is a nondecreasing function of P , it can be assumed without any loss of generality that the average power constraint is $\frac{1}{T}E[\text{tr}(\mathbf{X}\mathbf{X}^*)] = P$ instead of $\frac{1}{T}E[\text{tr}(\mathbf{X}\mathbf{X}^*)] \leq P$. The capacity function (in bits per dimension) admits the following Taylor series expansion:

$$C(P) = \dot{C}(0)P \log_2 e + \frac{1}{2}\ddot{C}(0)P^2 \log_2 e + o(P^2) \quad (61)$$

where $\dot{C}(0)$ and $\ddot{C}(0)$ are the first and second derivatives of $C(P)$ computed in nats per dimension. The notation and units introduced above for $C(P)$, $\dot{C}(0)$, and $\ddot{C}(0)$ will be used in the rest of this paper. The capacity per unit energy (in bits per joule)

is the reciprocal of $\frac{E_b}{N_{0\min}}$, and is equal to $\dot{C}(0)\log_2 e$ in the peak-constrained case, and either metric can be considered to be a measure of energy efficiency. Therefore, the notions of minimizing the energy per bit and maximizing the information rate per unit energy will be used interchangeably. The minimization of energy per bit is considered in Section IV-A. Note, however, that since this minimum occurs at a vanishing SNR, a fixed rate (in bits per second) of communication can be only achieved in the limit of infinite bandwidth. It is hence of interest to communicate at low but nonvanishing SNR and also do so in a bandwidth-efficient manner, which brings us to the notion of wideband slope introduced in [6].

The slope of the capacity function versus $\frac{E_b}{N_0}$ (also called the spectral efficiency function) in bits per second per hertz per 3 dB at zero spectral efficiency is defined as the wideband slope in [6] and was shown to be given in terms of $\dot{C}(0)$ and $\ddot{C}(0)$ as

$$\mathcal{S}_0 = \frac{2 \left[\dot{C}(0) \right]^2}{-\ddot{C}(0)}. \quad (62)$$

The motivation for considering the wideband slope as a performance metric is that, while achieving $\frac{E_b}{N_{0\min}}$ is desirable for energy efficiency, the rate of convergence of $\frac{E_b}{N_0}$ to $\frac{E_b}{N_{0\min}}$ as $P \rightarrow 0$ is also an important factor at low P , which in turn is closely tied to spectral efficiency. The higher the wideband slope, the greater is the spectral efficiency when operating at small but nonvanishing SNR. This point about the importance of the wideband slope was highlighted through several examples in the insightful work of [6]. An important example provided there was that of noncoherent communications with an input average-power constraint alone, and the wideband slope in this case was found to be $\mathcal{S}_0 = 0$ in contrast to that of coherent communication where it is positive. This result implies that to approach $\frac{E_b}{N_{0\min}}$, the bandwidth for reliable noncoherent communications becomes prohibitively large and the associated signaling scheme prohibitively peaky, and therefore no realistic (i.e., bandwidth-limited and peak-limited) scheme can achieve $\frac{E_b}{N_{0\min}}$.

In this work, the noncoherent MIMO channel is considered with a peak constraint on the input, in addition to the average-power constraint. It is shown that with the additional peak constraint, which is necessary for meaningful results at low SNR, there is a tradeoff between the minimum energy per bit and the wideband slope. This provides a far more detailed characterization of the wideband slope than if only the average-power constraint were imposed, and in particular it shows that it is possible to have $\mathcal{S}_0 > 0$ provided the peak constraint on the input is less than a certain constant. In the process, the $T+1$ point constellation is derived in Section IV-B from among constellations that achieve minimum energy per bit (or equivalently, $\dot{C}(0)$) that is optimal in wideband slope (or maximizes $\dot{C}(0)$), which interestingly, turns out to be STORM again. STORM is hence optimal in spectral efficiency in the wideband regime. Apart from providing fundamental limits on peak-limited MIMO noncoherent communications, our results and conclusions also temper the pessimistic conclusions that result from the consideration of noncoherent communication under just an average power constraint [6].

A. Achieving Minimum Energy Per Bit

In this subsection, the necessary and sufficient conditions for a constellation to achieve $\frac{E_b}{N_{0\min}}$ are derived. First, the following definition and lemma are needed from optimization theory [28], [29].

Definition 4: A function f is strictly quasi-concave over a convex set \mathcal{A} iff for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{A}$, and for $0 < \theta < 1$

$$f(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) > \min\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}. \quad (63)$$

Lemma 5: The global minimum of a strictly quasi-concave function f over a compact convex set \mathcal{A} is attained at a point $\mathbf{x} \in \mathcal{A}$ only if \mathbf{x} is an extreme point of \mathcal{A} .

Theorem 3: Consider a constellation \mathcal{C} with nonzero matrices $\{\mathbf{X}_i\}_{i=1}^{L-1}$ and respective probabilities $\{P_i\}_{i=1}^{L-1}$, and the zero matrix with probability P_0 . Let \mathcal{C} satisfy the average power constraint $\mathbb{E}[\text{tr}(\mathbf{X}\mathbf{X}^*)] = PT = E$ and the peak constraint (3) as in the peak-constrained problem. Then, \mathcal{C} achieves the capacity per unit energy as $P \rightarrow 0$ iff its constellation matrices and respective probabilities are of the following form:

$$\mathbf{X}_i = \sqrt{K} \mathbf{v}_i \mathbf{w}_i^*, \quad 1 \leq i \leq L-1 \quad (64)$$

$$\mathbf{X}_0 = \mathbf{0}_{T \times N_t} \quad (65)$$

$$\sum_{i=1}^{L-1} P_i = \frac{P}{KN_t} \quad (66)$$

$$P_0 = 1 - \frac{P}{KN_t} \quad (67)$$

where for each i , $\mathbf{v}_i \in \mathbb{C}^{T \times 1}$, $\mathbf{w}_i \in \mathbb{C}^{N_t \times 1}$, and $|\mathbf{v}_i \mathbf{w}_i^*|_{mn} = 1 \forall i, m, n$. The capacity per unit energy achieved by the above constellation is

$$N_r \cdot \left\{ 1 - \frac{\log(1 + KN_t T)}{KN_t T} \right\} \log_2 e \text{ bits per joule.} \quad (68)$$

Proof: Let the mutual information between \mathcal{C} and the output \mathbf{Y} be denoted as $I(P)$ (in nats per dimension). It is known from [30] that to achieve the capacity per unit energy, it is sufficient to use one symbol apart from the zero energy symbol. Therefore, our formulation, which assumes a discrete input with an arbitrary number of points, is without any loss of generality. The optimization problem that is to be solved is given as

$$\begin{aligned} & \max_{\{P_i\}_{i=0}^{L-1}, \{\mathbf{X}_i\}_{i=1}^{L-1}} \dot{I}(0) \\ & \text{subject to } \sum_{i=0}^{L-1} P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^*) = PT, \|\mathbf{X}_i\|_\infty \leq \sqrt{K}, \forall i \\ & \sum_{i=1}^{L-1} P_i = 1 - P_0, \quad P_i \geq 0 \forall i. \end{aligned} \quad (69)$$

A general formula for $\dot{I}(0)$ was derived in [6] and is given as

$$\dot{I}(0) = \lim_{P \rightarrow 0} \frac{\mathbb{E}_{\mathbf{X}} [D(P_{\mathbf{Y}|\mathbf{X}} \| P_{\mathbf{Y}|\mathbf{X}=0})]}{\mathbb{E}_{\mathbf{X}} [\text{tr}(\mathbf{X}\mathbf{X}^*)]}. \quad (70)$$

Since

$$\begin{aligned} & \max_{\{P_i\}_{i=0}^{L-1}, \{\mathbf{X}_i\}_{i=1}^{L-1}} \lim_{P \rightarrow 0} \frac{\mathbb{E}_{\mathbf{X}} [D(P_{\mathbf{Y}|\mathbf{X}} \| P_{\mathbf{Y}|\mathbf{X}=0})]}{\mathbb{E}_{\mathbf{X}} [\text{tr}(\mathbf{X}\mathbf{X}^*)]} \\ & \leq \lim_{P \rightarrow 0} \max_{\{P_i\}_{i=0}^{L-1}, \{\mathbf{X}_i\}_{i=1}^{L-1}} \frac{\mathbb{E}_{\mathbf{X}} [D(P_{\mathbf{Y}|\mathbf{X}} \| P_{\mathbf{Y}|\mathbf{X}=0})]}{\mathbb{E}_{\mathbf{X}} [\text{tr}(\mathbf{X}\mathbf{X}^*)]} \end{aligned} \quad (71)$$

an upper bound for the optimal value of the problem in (69) is

$$\begin{aligned} & \lim_{P \rightarrow 0} \max_{\{P_i\}_{i=0}^{L-1}, \{\mathbf{X}_i\}_{i=1}^{L-1}} \frac{\mathbb{E}_{\mathbf{X}} [D(P_{\mathbf{Y}|\mathbf{X}} \| P_{\mathbf{Y}|\mathbf{X}=0})]}{\mathbb{E}_{\mathbf{X}} [\text{tr}(\mathbf{X}\mathbf{X}^*)]}, \\ & \text{subject to } \sum_{i=0}^{L-1} P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^*) = PT, \|\mathbf{X}_i\|_{\infty} \leq \sqrt{K}, \forall i \\ & \sum_{i=1}^{L-1} P_i = 1 - P_0, \quad P_i \geq 0 \forall i. \end{aligned} \quad (72)$$

The objective function in (72) can be evaluated as

$$\begin{aligned} & \frac{\mathbb{E}_{\mathbf{X}} [D(P_{\mathbf{Y}|\mathbf{X}} \| P_{\mathbf{Y}|\mathbf{X}=0})]}{\mathbb{E}_{\mathbf{X}} [\text{tr}(\mathbf{X}\mathbf{X}^*)]} \\ & = N_r \cdot \left\{ 1 - \frac{\sum_{i=1}^{L-1} P_i \log \det(\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*)}{\sum_{i=1}^{L-1} P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^*)} \right\}. \end{aligned} \quad (73)$$

Consequently, the problem that needs to be solved is

$$\begin{aligned} & \min_{\{P_i\}_{i=0}^{L-1}, \{\mathbf{X}_i\}_{i=1}^{L-1}} \frac{\sum_{i=1}^{L-1} P_i \log \det(\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*)}{\sum_{i=1}^{L-1} P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^*)} \\ & \text{subject to } \sum_{i=0}^{L-1} P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^*) = PT, \|\mathbf{X}_i\|_{\infty} \leq \sqrt{K}, \forall i \\ & \sum_{i=1}^{L-1} P_i = 1 - P_0, \quad P_i \geq 0 \forall i. \end{aligned} \quad (74)$$

Relaxing the peak constraint, the optimal value of the problem in (74) over the signal constellation (but with the probabilities fixed) is lower-bounded by the optimal value of the problem

$$\begin{aligned} & \min_{\{\mathbf{X}_i\}_{i=1}^{L-1}, \{d_i\}_{i=1}^{L-1}, \{P_i\}_{i=0}^{L-1}} \frac{\sum_{i=1}^{L-1} P_i \log \det(\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*)}{\sum_{i=1}^{L-1} P_i d_i} \\ & \text{subject to } \sum_{i=0}^{L-1} P_i d_i = PT, \text{tr}(\mathbf{X}_i \mathbf{X}_i^*) = d_i, \\ & 0 \leq d_i \leq KN_t T, \forall i. \end{aligned} \quad (75)$$

The optimal values of problems (74) and (75) are the same iff the $\{\mathbf{X}_i\}_{i=1}^{L-1}$ that solves (75) also satisfies $\|\mathbf{X}_i\|_{\infty} \leq \sqrt{K}, \forall i$.

As in the PPAPR-constrained problem, the above problem can be solved as a two-stage optimization, where in the first stage, the probabilities $\{P_i\}_{i=0}^{L-1}$ are fixed and the constellation $\{\mathbf{X}_i\}_{i=1}^{L-1}$ is optimized. In the second step, the resulting objective function over is optimized over $\{P_i\}_{i=0}^{L-1}$.

Consider a fixed, feasible, but otherwise arbitrary $\{P_i\}_{i=0}^{L-1}$. It can be verified that for each i

$$\min_{\text{tr}(\mathbf{X}_i \mathbf{X}_i^*) = d_i} \log \det(\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*) = \log(1 + d_i) \quad (76)$$

is solved iff \mathbf{X}_i has unit rank.

Therefore, the problem in (75) can be rewritten as

$$\begin{aligned} & \min_{\{d_i\}_{i=1}^{L-1}, \{P_i\}_{i=0}^{L-1}} \frac{\sum_{i=1}^{L-1} P_i \log(1 + d_i)}{\sum_{i=1}^{L-1} P_i d_i}, \\ & \text{subject to } \sum_{i=1}^{L-1} P_i d_i = PT, \quad 0 \leq d_i \leq KN_t T, \forall i. \end{aligned} \quad (77)$$

Let $\mathbf{d} = [d_1 \ d_2 \ \dots \ d_{L-1}]^T$. Consider the set

$$\mathcal{A}_t = \left\{ \mathbf{d} : h(\mathbf{d}) = \frac{\sum_{i=1}^{L-1} P_i \log(1 + d_i)}{\sum_{i=1}^{L-1} P_i d_i} > t, d_i \geq 0 \forall i, t \geq 0 \right\}. \quad (78)$$

Since $\sum_{i=1}^{L-1} P_i \log(1 + d_i) - t \sum_{i=1}^{L-1} P_i d_i$ is strictly concave for every real t , the set \mathcal{A}_t is convex. Therefore, considering any two points $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{A}_t$ where $t = \min\{h(\mathbf{d}_1), h(\mathbf{d}_2)\}$ and using Definition 4, $\sum_{i=1}^{L-1} P_i \log(1 + d_i) / \sum_{i=1}^{L-1} P_i d_i$ is a strictly quasi-concave function of \mathbf{d} . Hence, from Lemma 5, the solution of (77) is achieved at a vertex of the constraint set. Using Lemma 4, each vertex of the constraint set consists of $L - 1$ entries that are either $KN_t T$ or 0, and exactly one entry c such that $0 \leq c \leq KN_t T$.

It can therefore be assumed, without loss of generality, that the optimal \mathbf{d} and the corresponding probabilities are

$$\begin{aligned} \mathbf{d} &= \begin{bmatrix} KN_t T & \dots & KN_t T & c & 0 \end{bmatrix}^T, \quad 0 \leq c \leq KN_t T \\ \mathbf{P} &= [P_1 \ \dots \ P_M \ P_c \ P_0]^T \end{aligned} \quad (80)$$

where, for convenience, the symbol M is introduced to denote the number of entries in \mathbf{d} that are equal to $KN_t T$. Since the objective function is a symmetric function of \mathbf{d} , the specific arrangement of the entries is immaterial. Using this structure for \mathbf{d} , the problem in (77) can be rewritten and bounded from below as

$$\begin{aligned} & \min_{c, \{P_i\}_{i=0}^{L-1}} \frac{\sum_{i=1}^M P_i \log(1 + KN_t T) + P_c \log(1 + c)}{\sum_{i=1}^M P_i KN_t T + c P_c} \\ & \text{subject to } 0 \leq c \leq KN_t T, \sum_{i=1}^M P_i KN_t T + c P_c = PT, \\ & \sum_{i=1}^M P_i = 1 - P_0 \end{aligned} \quad (81)$$

$$\begin{aligned} & \geq \min_{\substack{c, \{P_i\}_{i=0}^{L-1} \\ 0 \leq c \leq KN_t T}} \frac{\sum_{i=1}^M P_i \log(1 + KN_t T) + P_c \log(1 + c)}{\sum_{i=1}^M P_i KN_t T + c P_c}. \end{aligned} \quad (82)$$

The problem in (82) is easily seen to be the minimization of a strictly quasi-concave function over c . Therefore, the solution has to be among the vertices of $0 \leq c \leq KN_t T$, i.e., either $c = 0$ or $c = KN_t T$. Notice that with either choice of c , the objective function is $\frac{\log(1+KN_t T)}{KN_t T}$, and is independent of $\{P_i\}_{i=0}^{L-1}$. Therefore, the upper bound on the optimal value of the problem in (69) is

$$N_r \cdot \left\{ 1 - \frac{\log(1+KN_t T)}{KN_t T} \right\}. \quad (83)$$

Since $d_i = KN_t T \forall i$, for equality to hold in the inequality leading to (75), it is necessary and sufficient that the nonzero matrices $\{\mathbf{X}_i\}_{i=1}^{L-1}$ be of the form

$$\mathbf{X}_i = \sqrt{K} \mathbf{v}_i \mathbf{w}_i^* \forall i \quad (84)$$

where $\mathbf{v}_i \in \mathbb{C}^{T \times 1}$, $\mathbf{w}_i \in \mathbb{C}^{N_t \times 1}$ are such that $|\mathbf{v}_i \mathbf{w}_i^*|_{mn} = 1 \forall i, m, n$. By substituting (84) in (69), a lower bound on the optimal value of (69) is obtained, which coincides with the upper bound in (83), implying that (83) is the optimal value of the problem in (69). From the power constraint, $\sum_{i=1}^{L-1} P_i = \frac{P}{KN_t}$ must be true and $P_0 = 1 - \frac{P}{KN_t} > 0$. Therefore, it can be concluded that

$$\dot{C}(0) = N_r \cdot \left\{ 1 - \frac{\log(1+KN_t T)}{KN_t T} \right\}. \quad (85)$$

□

Note that the capacity per unit energy in (68) is independent of the number of points L . In particular, it can be achieved with a two-point constellation.

Corollary 1: The following two-point constellation achieves the capacity per unit energy as the average power $P \rightarrow 0$:

$$(\mathbf{X}_1, P_1) = \left(\sqrt{K} \mathbf{v} \mathbf{w}^*, \frac{P}{KN_t} \right) \quad (86)$$

$$(\mathbf{X}_2, P_2) = \left(\mathbf{0}_{T \times N_t}, 1 - \frac{P}{KN_t} \right) \quad (87)$$

where \mathbf{v} and \mathbf{w} are column vectors such that $|\mathbf{v} \mathbf{w}^*|_{mn} = 1 \forall m, n$. □

The above two-point constellation is referred to as MIMO-OOK (on-off keying). This constellation can also be obtained directly through a simplified general formula for the capacity per unit energy derived in [30]. It turns out that the simplified formula in [30] can be evaluated using similar techniques to those used in the proof of Theorem 3, and is also a more direct approach than the derivation of the capacity per unit energy in [31]. For the sake of completeness, it is given in Appendix C.

Clearly, Theorem 3 implies that there is a large class of constellations which achieve $\frac{E_b}{N_{0 \min}}$. For instance, the cardinality

can be any $L \geq 2$. Moreover, only the sum of probabilities of the nonzero points is constrained to be $\frac{P}{KN_t}$, while the individual probabilities can be arbitrary. Further, there is no restriction on the relationship between \mathbf{X}_i and \mathbf{X}_j , $\forall j \neq i$. In particular, \mathbf{X}_i can be taken to be all equal to a unit rank matrix \mathbf{X} with elements of equal magnitude (equal to \sqrt{K}) for all $i = 1, 2, \dots, L-1$. In this case, the nonzero points would coincide and become one nonzero point with probability $\frac{P}{KN_t}$, thereby reducing to the two-point MIMO-OOK constellation of Corollary 1.

B. Maximizing the Wideband Slope

A key insight provided by [6] is that even though different schemes may achieve $\frac{E_b}{N_{0 \min}}$, an analysis of their wideband slopes could reveal vast differences in the rate of growth of their energy efficiencies around $\frac{E_b}{N_{0 \min}}$, and therefore differentiates their spectral efficiencies. The wideband slope, which is the measure of spectral efficiency at low but nonvanishing SNR, is therefore critical in the analysis of wideband channels. Our next aim is therefore, to optimize the wideband slope over constellations which achieve $\frac{E_b}{N_{0 \min}}$. The next theorem provides a formula for the wideband slope \mathcal{S}_0 when evaluated for an arbitrary generalized OOK constellation.

Theorem 4: Consider a constellation \mathcal{C} with nonzero matrices $\{\mathbf{X}_i\}_{i=1}^M$ and respective probabilities $\{P_i\}_{i=1}^M$, and the zero matrix with probability P_0 . Then we get (88) at the bottom of the page.

Proof: See Appendix D.

The following corollary indicates a fundamental limitation in approaching the capacity per unit energy for a constellation of arbitrary cardinality.

Corollary 2: Consider a constellation \mathcal{C} with nonzero matrices $\{\mathbf{X}_i\}_{i=1}^M$ and respective probabilities $\{P_i\}_{i=1}^M$, and the zero matrix with probability P_0 . Let \mathcal{C} satisfy the average and peak power constraints in the statement of Theorem 3. Suppose \mathcal{C} achieves the capacity per unit energy. Then the wideband slope \mathcal{S}_0 is 0 when $KN_t T > 1$.

Proof: Since \mathcal{C} achieves the capacity per unit energy, it satisfies the necessary conditions stated in Theorem 3. From Theorem 4, the wideband slope is nonzero only when the matrix

$$\mathbf{I} - \mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_j \mathbf{X}_j^* \quad (89)$$

is positive definite for all pairs i, j . The proof of the corollary follows when the necessary conditions for achieving the capacity per unit energy in Theorem 3 are substituted in (89) and simplified. □

Theorem 5: Among all constellations of Theorem 3 which achieve $\frac{E_b}{N_{0 \min}}$, with $T+1$ points, STORM has the maximum wideband slope.

$$\mathcal{S}_0 = \begin{cases} \frac{2}{T} \frac{N_r^2 \left(\sum_{i=1}^M P_i \text{tr}(\mathbf{X}_i \mathbf{X}_i^*) - \sum_{i=1}^M P_i \log \det(\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*) \right)^2}{\sum_{i=1}^M \frac{P_i^2}{(1-P_0)^2} \frac{1}{|\mathbf{I} - \mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_i \mathbf{X}_i^*|^{N_r}} + \sum_{i \neq j} \frac{P_i P_j}{(1-P_0)^2} \frac{1}{|\mathbf{I} - \mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_j \mathbf{X}_j^*|^{N_r}} - 1}, & \text{if } \mathbf{I} - \mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_j \mathbf{X}_j^* \text{ is positive definite } \forall i, j \\ 0, & \text{otherwise.} \end{cases} \quad (88)$$

Proof: Since the constellations under consideration achieve $\frac{E_b}{N_{0\min}}$, the numerator in (88) is a fixed constant. Further, given the necessary conditions for the constellation to achieve $\frac{E_b}{N_{0\min}}$, the denominator of the wideband slope can be simplified as

$$\sum_{i=1}^M \frac{P_i^2}{(1-P_0)^2} \frac{1}{(1-K^2 N_t^2 T^2)^{N_r}} + \sum_i \sum_{j \neq i} \frac{P_i P_j}{(1-P_0)^2} \frac{1}{|\mathbf{I} - \mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_j \mathbf{X}_j^*|^{N_r}} - 1 \quad (90)$$

where the matrices $\{\mathbf{X}_i\}_{i=1}^M$ are of unit rank with entries of equal magnitude \sqrt{K} , and $KN_t T < 1$ (to ensure that $\mathbf{I} - \mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_j \mathbf{X}_j^*$ is positive semidefinite $\forall i, j$). Clearly, (90) is minimized when there exist rank-one matrices $\{\mathbf{X}_i\}_{i=1}^M$ such that $\mathbf{X}_j^* \mathbf{X}_i = \mathbf{0} \forall i, j \neq i$. Such a set exists for $M \leq T$, and is denoted by $\mathbf{X}_i = \sqrt{K} \mathbf{v}_i \mathbf{w}_i^* \forall i$, where the definitions for \mathbf{v}_i and \mathbf{w}_i are the same as in Theorem 2. The problem that needs to be solved is thus as shown in (91) at the bottom of the page. The objective function in (91) can be easily shown to be a Schur-convex function [32] of $[P_1 P_2 \dots P_M]$. Hence, the minimum occurs when each of the probabilities $\{P_i\}_{i=1}^M$ is equal to $\frac{1-P_0}{M}$. The optimal value of (91) is therefore

$$\frac{1}{M} \left\{ \frac{1}{(1-K^2 N_t^2 T^2)^{N_r}} - 1 \right\}. \quad (92)$$

Clearly, M has to be made as large as possible, but to ensure achievability of the optimal value in (91), it can be no greater than $T+1$. Therefore, set $M = T+1$. Evidently, the solution to (91) when $M > T+1$ would provide an upper bound on the maximum wideband slope. \square

Theorem 5 establishes the optimality of STORM among $T+1$ -point constellations in the peak-constrained case. This means that STORM is spectrally most efficient among all $T+1$ (or fewer) point constellations that achieve maximum capacity per unit energy in the low-SNR regime.

The following corollary provides the wideband slopes of MIMO-OOK and STORM.

Corollary 3: The wideband slopes of MIMO-OOK and STORM are respectively

$$\mathcal{S}_0^{OOK} = \begin{cases} \frac{2}{T} \frac{N_r^2 (KN_t T - \log(1+KN_t T))^2}{(1-K^2 N_t^2 T^2)^{N_r} - 1}, & \text{if } KN_t T < 1 \\ 0, & \text{if } KN_t T \geq 1. \end{cases} \quad (93)$$

$$\mathcal{S}_0^{STORM} = \begin{cases} 2 \frac{N_r^2 (KN_t T - \log(1+KN_t T))^2}{(1-K^2 N_t^2 T^2)^{N_r} - 1}, & \text{if } KN_t T < 1 \\ 0, & \text{if } KN_t T \geq 1. \end{cases} \quad (94)$$

Proof: The wideband slopes follow by substituting the MIMO-OOK and STORM constellations in the result of Theorem 4. \square

C. Remarks

Since STORM was obtained as the optimal constellation even in the PAPR constrained case, many of the remarks on STORM following Theorem 2 and in Section III-A apply even to the peak-constrained case. Here we only state new insights pertinent to the peak-constrained case.

1. From (68), it is seen that $\lim_{K \rightarrow \infty} \dot{C}(0) = N_r$. Therefore, for asymptotically large peak powers, the well-known result on the capacity per unit energy with only an average power constraint [6] which is common to both coherent and noncoherent MIMO channels, is recovered. Indeed, when $N_r = 1$, we obtain the minimum energy to transmit one bit of information to be -1.59 dB, which is a classical result. By relaxing the peak constraint, STORM can be seen to be optimal even for the case when there is merely an average power constraint (or with respect to infinite bandwidth capacity).

2. When the signals are just subject to an average power constraint, it is shown in [6] that $\mathcal{S}_0 = 0$ for the noncoherent MIMO channel. Therefore, signals whose energy per bit approaches $\frac{E_b}{N_{0\min}}$ would have to have bandwidths that become prohibitively large. However, when there is an additional peak-power constraint K which is a fixed constant, and for the case when the normalized peak power $KN_t T < 1$, Corollary 3 shows that \mathcal{S}_0 is strictly positive. Hence, it is realistic to design signals that achieve the $\frac{E_b}{N_{0\min}}$ in this scenario for low but nonvanishing SNR. Similar insights were also noted in [12] but in the simpler context of the SISO Rician-fading channel with unit block length under peak and average power constraints.

3. While both MIMO-OOK and STORM achieve $\frac{E_b}{N_{0\min}}$, according to Corollary 3, the wideband slope of STORM is higher by a factor of T . This means that at a certain energy per bit and for the same transmission rate, and as $\text{SNR} \rightarrow 0$, the bandwidth needed by STORM for the same spectral efficiency is less than that of MIMO-OOK by a factor of T . Given typical values of the coherence time T , this higher spectral efficiency of STORM can translate into huge savings. To give a sense of the significant gains, Figs. 1 and 2 plot the spectral efficiency versus the energy per bit for STORM and MIMO-OOK.

4. Figs. 3 and 4 plot the energy per bit and wideband slope of STORM versus the normalized peak power $KN_t T$, for different values of N_r . As the normalized peak power increases, it is seen that the $\frac{E_b}{N_{0\min}}$ decreases. This is expected as peakier signaling is more energy efficient. However, as the normalized peak power gets close to 1, the wideband slope approaches 0. In fact, the wideband slope attains its maximum at an intermediate value between 0 and 1 (say, $KN_t T = c^*$). Since for any

$$\min_{\substack{\{P_i\}_{i=1}^M \\ \sum_{i=1}^M P_i = 1 - P_0, \sum_{i=1}^M P_i = P / KN_t}} \sum_{i=1}^M \frac{P_i^2}{(1-P_0)^2} \left\{ \frac{1}{(1-K^2 N_t^2 T^2)^{N_r}} - 1 \right\}. \quad (91)$$

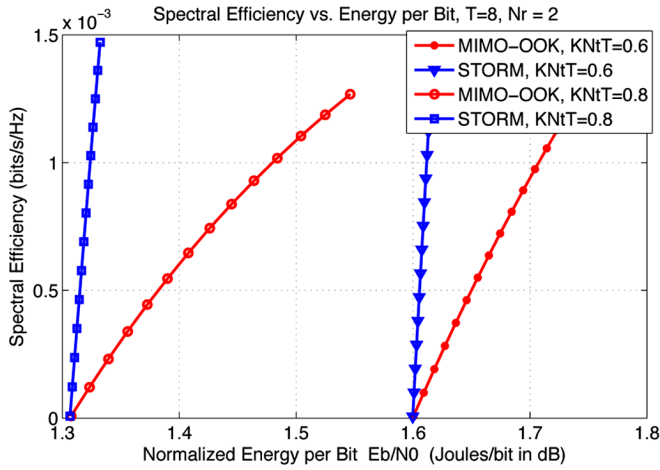


Fig. 1. Plot of spectral efficiency versus energy per bit of STORM using the second-order approximation of $I(P)$ in (95), for different values of $KN_t T$.

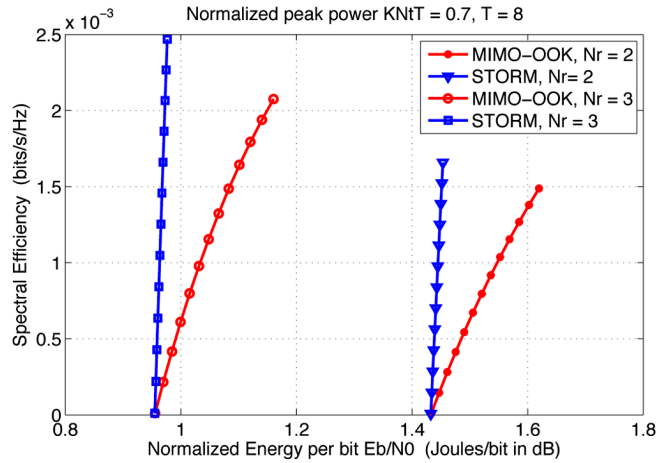


Fig. 2. Plot of spectral efficiency versus energy per bit of STORM using the second-order approximation of $I(P)$ in (95), for different values of N_r .

point in the region $0 \leq KN_t T \leq c^*$ there is a point corresponding to $c^* \leq KN_t T \leq 1$ with lower $\frac{E_b}{N_{0\min}}$ and the same wideband slope, it makes most sense to operate in the region $c^* \leq KN_t T \leq 1$. Assuming only an average power constraint, the analysis in [6] shows that $\mathcal{S}_0 = 0$ for noncoherent communications. The scheme that achieves the $\frac{E_b}{N_{0\min}}$ has the nonzero signals migrating to ∞ in amplitude as $P \rightarrow 0$. The results in [6] show in effect that it is unrealistic to realize the peak-unconstrained minimum energy per bit (STORM having zero wideband slope for all $KN_t T \geq 1$ is clearly a stronger statement). Under realistic assumptions on the peak constraint, however, it has been shown here that $\mathcal{S}_0 > 0$ is possible when $KN_t T < 1$. Moreover, a sharp characterization is provided which shows that there is a tradeoff between $\frac{E_b}{N_{0\min}}$ and \mathcal{S}_0 for STORM in the region $c^* \leq KN_t T \leq 1$.

5. For the same number of bits transmitted reliably per joule at low SNR, MIMO-OOK requires an operating SNR which is $10 \log_{10} T$ decibels smaller than that of STORM. This can be seen from the fact that the wideband slope of STORM is T times that of MIMO-OOK and that mutual information per joule is given as

$$I(P)/P = \dot{I}(0)(\log_2 e) + \frac{1}{2} \ddot{I}(0)(\log_2 e)P + o(P). \quad (95)$$

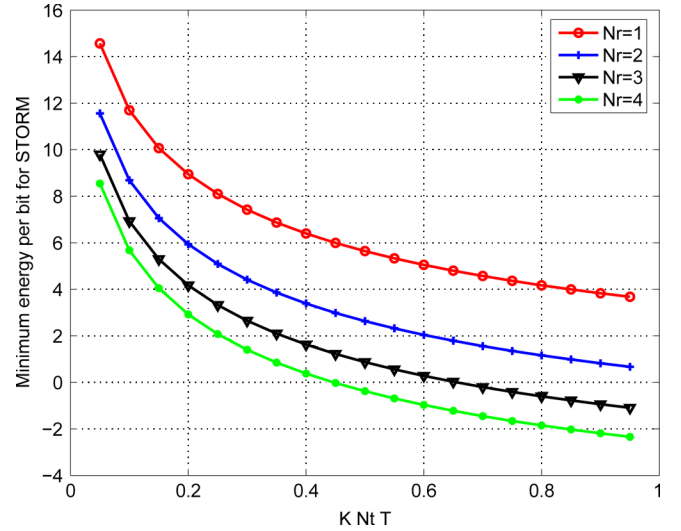


Fig. 3. Energy per bit (decibels) of STORM versus $KN_t T$ for different N_r .

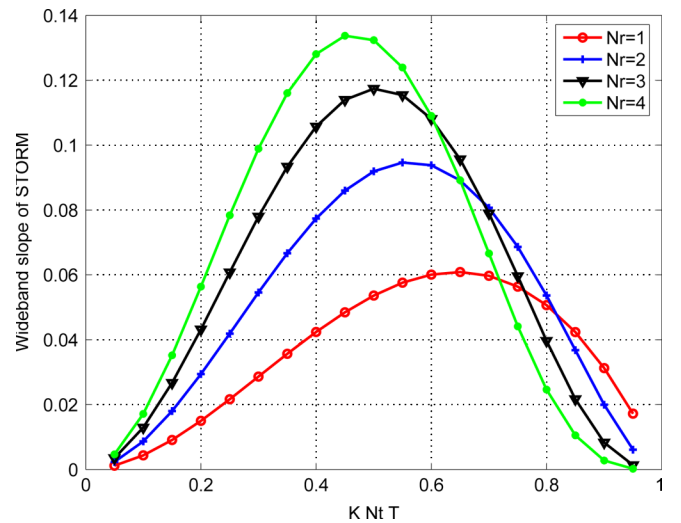


Fig. 4. Wideband slope of STORM versus $KN_t T$ for different N_r .

and the wideband slope is $\mathcal{S}_0^I = -\frac{2[\dot{I}(0)]^2}{\ddot{I}(0)}$. Now, since the peak power is a fixed constant, this implies that the PAPR of MIMO-OOK at any small but nonvanishing SNR would be greater than that of STORM by a factor of T . Since in the low-SNR regime, peakiness of the signal constellations is a crucial factor, using STORM can potentially result in large reductions in the required PAPR and facilitate implementation. These large savings are illustrated in Fig. 5, where the approximation of $\frac{I(P)}{P}$ versus P is plotted for STORM and MIMO-OOK. In the example shown, the convergence to the capacity per unit energy is faster for STORM by a factor of $10 \log_{10} T = 9$ dB relative to MIMO-OOK.

6. It has been shown in Corollary 2 that whenever $KN_t T > 1$, the wideband slope is 0. Therefore, even though the noncoherent capacity per unit energy is $N_r \log_2 e$ bits per joule, it is prohibitively expensive (in terms of bandwidth) to reliably transmit at any rate more than the peak-constrained capacity per unit energy evaluated at $KN_t T = 1$ which from (68) is $N_r(\log_2 e - 1)$ bits per joule. Hence, the capacity per unit energy at $KN_t T = 1$ can be taken to be the realistic limit for noncoherent MIMO communication. Note that this limit is also N_r bits per joule smaller

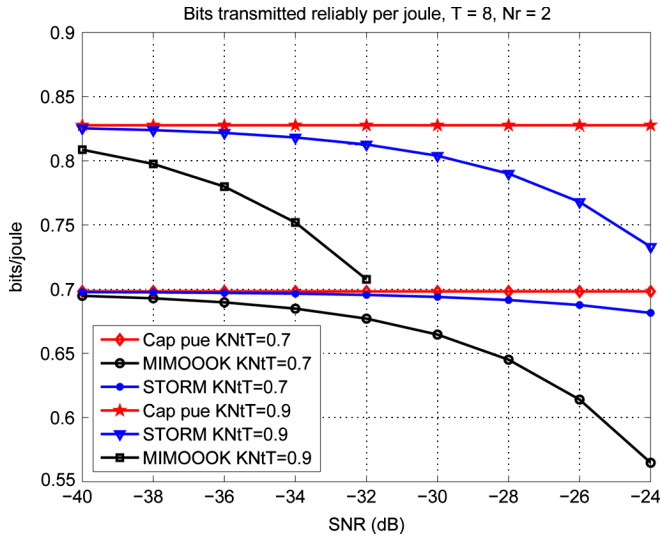


Fig. 5. Plot comparing first-order approximation of $I(P)/P$ in (95) vs. P for STORM and MIMO-OOK for different values of KN_tT .

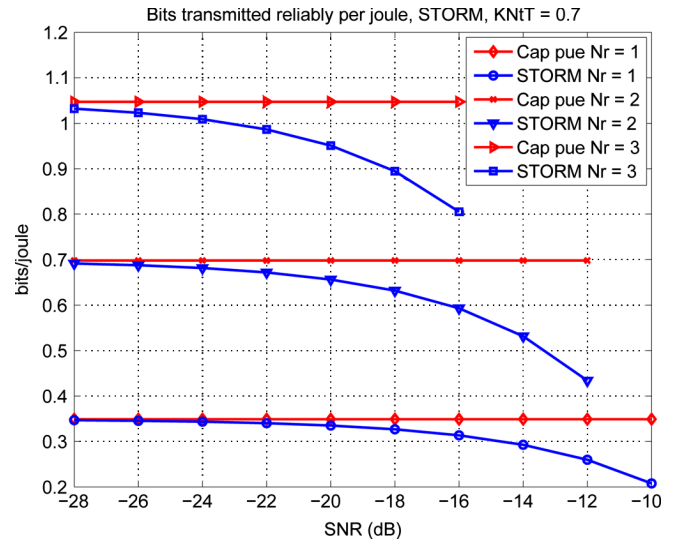


Fig. 7. Plot of first-order approximation of $I(P)/P$ in (95) versus P for STORM, for different values of N_r .

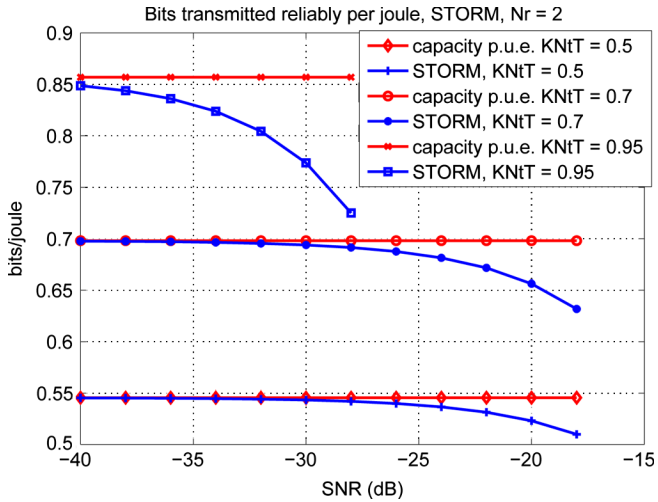


Fig. 6. Plot of first-order approximation of $I(P)/P$ in (95) versus P for STORM, and its convergence to the capacity per unit energy, for different values of KN_tT .

than the coherent capacity per unit energy. Since the analysis of the noncoherent channel neither assumes any particular scheme for channel estimation nor does it ignore the resources for (implicit) channel estimation, the realistic capacity per unit energy of $N_r(\log_2 e - 1)$ bits per joule can be argued as being more fundamental than the coherent capacity per unit energy of $N_r \log_2 e$ bits per joule. The difference between the two can be thought of as the fundamental or minimal cost of (implicit) channel estimation.

7. The dependence of $\frac{E_b}{N_0 \min}$ on K , N_t , and T is only through the product KN_tT . So, increasing one or more these quantities has the effect of lowering $\frac{E_b}{N_0 \min}$. However, this effect is beneficial when $KN_tT < c^*$ and beyond that the tradeoff between energy efficiency and bandwidth efficiency is quantified here that allows a designer to choose a suitable operating point. To illustrate this point, Fig. 6 plots the approximation of $I(P)/P$ versus P for different values of KN_tT . It is evident from Fig. 6, that even as KN_tT gets close to one, the bits re-

quired to transmit reliably converges to the capacity per unit energy at much smaller SNRs (and hence larger bandwidths). Since the PAPR of STORM at SNR P is KN_t/P , it is interesting to note that when KN_tT is fixed, increasing T decreases the PAPR required for the same energy per bit which is an advantage in practice. Increasing N_t with KN_tT fixed, decreases K , and therefore reduces the peak-power per antenna and time slot (though not changing the PAPR), which may also be helpful in practice.

8. An interesting observation from Figs. 3 and 4 is that using more receive antennas always lowers $\frac{E_b}{N_0 \min}$, while it does not always increase the wideband slope. Fig. 7 illustrates that while the approximation of $I(P)/P$ increases with N_r in general, the convergence to the capacity per unit energy occurs more slowly and hence a lower SNR is needed to operate close to it as N_r increases.

9. Even though the optimal scheme for a cardinality more than $T+1$ is yet unknown, STORM offers a concrete solution whose structure is also simple and practical. In [6], the positive impact on the wideband slope of using constellations with cardinality greater than two is illustrated via several contexts other than under the noncoherent assumption. Even so, following [30] and due to analytical convenience, many recent papers [12], [31], [33] in noncoherent communications focus on the two point ON-OFF scheme to achieve the capacity per unit energy. The results in this section demonstrate that there are compelling reasons to look beyond the two point ON-OFF scheme in the low-SNR regime.

10. Recently, the authors of [34]–[37] have investigated the possibility of channel coherence length scaling with SNR, so as to diminish the cost of acquiring channel knowledge. It should be interesting to pose and solve the optimization problems of this work under such scenarios.

V. CONCLUSION

Two important problems are posed on reliable communications over noncoherent MIMO spatially i.i.d. Rayleigh-fading

channels at low SNR. In both formulations, an average-power constraint on the input and a natural per-antenna, per-time slot peak-power constraint are assumed. In the first problem formulation, the peak-power to average-power ratio is held fixed (PPAPR-constrained) and the mutual information which grows as $O(\text{SNR}^2)$ is maximized up to second order jointly over input signal matrices and their respective probabilities, when the cardinality of the constellation is no greater than $T + 1$ (T is the coherence block length). In the second problem formulation (peak constrained), the peak power is a fixed constant independent of SNR. Here, necessary and sufficient conditions for a constellation of any cardinality to achieve the minimum energy per bit are derived. Over the set of all $T + 1$ -point constellations which achieve the minimum energy per bit, the second-order behavior of mutual information is optimized. The resulting constellations are both first- and second-order optimal among all $T + 1$ -point constellations. Both the PPAPR-constrained and peak-constrained problems result in finite-dimensional nonconvex optimization problems. Even so, they admit elegant solutions in closed form, which are identical in both formulations. This common solution is referred to as space-time orthogonal rank-one modulation (STORM), and it provides several new insights on noncoherent communications at low SNR.

In the PPAPR-constrained case, it is shown that the $T + 1$ -point STORM is near-optimal with respect to the maximum mutual information up to second order with unconstrained cardinality even for modest values of T and PAPR. Therefore, there is not much to be gained by using more than $T + 1$ points in the PPAPR-constrained case. In the peak-constrained case, the approach in this work provides a sharp characterization of the first- and second-order behavior of noncoherent MIMO capacity, that also sheds light on the cost of implicit estimation of channel state information in the low-SNR regime. The energy per bit and the wideband slope achieved by STORM also reveals a fundamental energy-versus-bandwidth efficiency tradeoff that enables the determination of the operating (low) SNR and peak power most suitable for a given application. Moreover, while the more conventional MIMO on-off keying (OOK) also achieves the minimum energy per bit, STORM has a wideband slope that is T times greater which translates into an increase in bandwidth efficiency (or a decrease in the PAPR) by a factor of T in the wideband regime. Given typical values of the coherence block length T , these gains are potentially huge.

APPENDIX

A. Proof of Nonconvexity

A simple argument is given to show that (25) is a nonconvex optimization problem.

We need the following definition of matrix convexity from [29].

Definition 5: A function $f : \mathfrak{R}^{n \times n} \rightarrow \mathfrak{R}^{m \times m}$ is matrix convex with respect to matrix inequality if for any positive semidefinite $\mathbf{X}_1, \mathbf{X}_2$ and for any $\theta \in [0, 1]$

$$f(\theta \mathbf{X}_1 + (1 - \theta) \mathbf{X}_2) \preceq \theta f(\mathbf{X}_1) + (1 - \theta) f(\mathbf{X}_2). \quad (96)$$

Since $\{\mathbf{X}_i\}_{i=1}^L$ is a set of complex matrices, the optimization over the signals amounts to an equivalent joint optimization over the real and imaginary parts of \mathbf{X}_i given by $\mathbf{X}_i = \hat{\mathbf{X}}_i + j\check{\mathbf{X}}_i, \forall i$. In order to show that this joint optimization is nonconvex, we will consider the contour given by $\check{\mathbf{X}}_i = \mathbf{0}, \forall i$. With the imaginary parts being zero, the function in (25) becomes $\sum_i P_i (1 - P_i) \text{tr}(\hat{\mathbf{X}}_i \hat{\mathbf{X}}_i^* \hat{\mathbf{X}}_i \hat{\mathbf{X}}_i^*)$

It can be seen that $g(\hat{\mathbf{X}}) = \hat{\mathbf{X}} \hat{\mathbf{X}}^*$ is matrix-convex over $\hat{\mathbf{X}}$, and $h(\mathbf{A}) = \text{tr}(\mathbf{A} \mathbf{A}^*)$ is a nondecreasing convex function over positive semidefinite matrices \mathbf{A} . Therefore, the composition $f(\hat{\mathbf{X}}) = h \circ g = \text{tr}(\hat{\mathbf{X}} \hat{\mathbf{X}}^* \hat{\mathbf{X}} \hat{\mathbf{X}}^*)$ is a convex function over $\hat{\mathbf{X}}$ [29]. Further, since $\text{tr}(\hat{\mathbf{X}} \hat{\mathbf{X}}^*)$ and $\|\hat{\mathbf{X}}\|_\infty$ are convex functions of $\hat{\mathbf{X}}$ [29], the constraints $\sum_i P_i \text{tr}(\hat{\mathbf{X}}_i \hat{\mathbf{X}}_i^*) \leq E$ and $\|\hat{\mathbf{X}}_i\|_\infty \leq \sqrt{K}$ are convex sets in $\{\hat{\mathbf{X}}_i\}_{i=1}^L$. For an arbitrary but fixed set of probabilities $\{P_i\}_{i=1}^L$, the objective function is convex in $\{\hat{\mathbf{X}}_i\}_{i=1}^L$, while the constraint set is the intersection of convex sets and is hence convex. Therefore, the problem of optimizing (25) over $\{\hat{\mathbf{X}}_i\}_{i=1}^L$ is a convex maximization problem and not a convex optimization problem. Since for a fixed $\{P_i\}_{i=1}^L$, the problem of optimizing over $\{\mathbf{X}_i\}_{i=1}^L$ is a nonconvex optimization problem for the imaginary parts of \mathbf{X}_i fixed, the joint optimization over $\{P_i\}_{i=1}^L$ and $\{\mathbf{X}_i\}_{i=1}^L$ is also nonconvex.

B. A Low-Complexity Block Decoder

In some applications, decoding of a block of symbols at a time may be required. This need arises for instance in uncoded systems, where there is no coding across blocks. Another possibility is when there is coding across blocks, but hard-decision decoding is employed at the receiver so that the blocks of symbols are first decoded via the maximum *a posteriori* probability (MAP) rule following which the outer code is decoded. In all such cases, we show in this section that the optimal MAP decoding of STORM can be simplified using FFT or FHT algorithms.

Consider the $T + 1$ -point STORM as described in (11) and (12). Let the received signal matrix be $\mathbf{R} \in \mathbb{C}^{T \times N_r}$. The optimal MAP rule to decode a block at the receiver is

$$\begin{aligned} \hat{j} &= \max_j P_j p(\mathbf{R} | \mathbf{X}_j) \\ &= \max_j P_j \frac{\exp\left\{-\text{tr}\left(\mathbf{R}^* (\mathbf{I}_T + \mathbf{X}_j \mathbf{X}_j^*)^{-1} \mathbf{R}\right)\right\}}{\pi^{TN_r} |\mathbf{I}_T + \mathbf{X}_j \mathbf{X}_j^*|^{N_r}}. \end{aligned} \quad (97)$$

For convenience, we will first find the maximum in (98) among the nonzero signal matrices, and then compare it with the metric for the zero matrix. Substituting STORM that is defined with permutation matrix \mathbf{P} , we get that the maximum metric among nonzero matrices is

$$\max_{i=1, \dots, T} \frac{E}{KN_t T^2} \frac{\exp\left\{-\text{tr}\left(\mathbf{Y}^* (\mathbf{I}_T + KN_t \mathbf{v}_i \mathbf{v}_i^*)^{-1} \mathbf{Y}\right)\right\}}{\pi^{TN_r} |\mathbf{I}_T + KN_t \mathbf{v}_i \mathbf{v}_i^*|^{N_r}} \quad (99)$$

where $\mathbf{Y} = \mathbf{P}^* \mathbf{R}$ is a sufficient statistic, which is simply the received matrix with the permutation removed. The term

$(\mathbf{I}_T + KN_t \mathbf{v}_i \mathbf{v}_i^*)^{-1}$ can be simplified by applying the Woodbury's identity, i.e., using

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1} \mathbf{B} (\mathbf{C}^{-1} + \mathbf{DA}^{-1} \mathbf{B})^{-1} \mathbf{DA}^{-1}.$$

Also, using the identity $|\mathbf{I} + \mathbf{AB}| = |\mathbf{I} + \mathbf{BA}|$, (99) becomes

$$\max_{i=1, \dots, T} \frac{E \exp\{-\text{tr}(\mathbf{Y}\mathbf{Y}^*)\}}{\pi^{TN_r} KN_t T^2 (1 + KN_t T)^{N_r}} \cdot \exp\left\{\frac{KN_t}{1 + KN_t T} \text{tr}(\mathbf{Y}^* \mathbf{v}_i \mathbf{v}_i^* \mathbf{Y})\right\}. \quad (100)$$

Clearly, among the nonzero constellation matrices, the MAP metric is maximized when $\|\mathbf{Y}^* \mathbf{v}_i\|^2$ is maximized. Let \mathbf{V} be the T dimensional DFT or Hadamard matrix. Then each row of the matrix $\mathbf{Y}^* \mathbf{V}$ would represent the DFT or Hadamard transform of the corresponding row of \mathbf{Y}^* . The nonzero constellation matrix with the maximum MAP metric would therefore correspond to the column of $\mathbf{Y}^* \mathbf{V}$ with the maximum l_2 -norm. The N_r DFTs or Hadamard transforms involved can be efficiently computed using fast algorithms (FFTs and FHTs). Now, the metric corresponding to the zero matrix would be

$$\left(1 - \frac{E}{KN_t T}\right) \frac{\exp(-\text{tr}(\mathbf{Y}\mathbf{Y}^*))}{\pi^{TN_r}}. \quad (101)$$

Since this is a constant for a given received signal, we can divide the metric in (100) by (101) and then take the natural logarithm of the resulting expression so that

$$\Omega_i = \ln \left\{ \frac{E}{T(KN_t T - E)(1 + KN_t T)^{N_r}} + \frac{KN_t}{1 + KN_t T} \text{tr}(\mathbf{Y}^* \mathbf{v}_i \mathbf{v}_i^* \mathbf{Y}) \right\}. \quad (102)$$

Now letting $i = \arg \max_{k=1, \dots, T} \|\mathbf{Y}^* \mathbf{v}_k\|^2$, the final simplified decoding rule can be given as

$$\hat{j} = \begin{cases} i, & \text{if } \Omega_i \geq 0 \\ T + 1, & \text{if } \Omega_i < 0. \end{cases} \quad (103)$$

C. Derivation of MIMO-OOK

Theorem 6: The capacity per unit energy (in nats per joule) for the i.i.d. MIMO block Rayleigh-fading channel with a peak-power constraint on the input signal $\|\mathbf{X}\|_\infty \leq \sqrt{K}$ is

$$\dot{C}(0) = N_r \left(1 - \frac{\log(1 + KN_t T)}{KN_t T}\right), \quad (104)$$

and is achieved as $P \rightarrow 0$ by the two-point constellation given as

$$(\mathbf{X}_1, P_1) = \left(\sqrt{K} \mathbf{v} \mathbf{w}^*, \frac{P}{KN_t}\right) \quad (105)$$

$$(\mathbf{X}_2, P_2) = \left(\mathbf{0}_{T \times N_t}, 1 - \frac{P}{KN_t}\right) \quad (106)$$

where $\mathbf{v} \in \mathbb{C}^{T \times 1}$, $\mathbf{w} \in \mathbb{C}^{N_t \times 1}$ and $|\mathbf{v}\mathbf{w}^*|_{mn} = 1 \forall m, n$.

Proof: From [30], it is known that to achieve the channel capacity per unit energy, it is enough to transmit one nonzero symbol, given in (105), apart from the symbol $\mathbf{0}$. Since we are dealing with a memoryless, discrete, and matrix input channel (1) with the cost function given by $b(\mathbf{X}) = \text{tr}(\mathbf{X}\mathbf{X}^*)$, the capacity per unit energy under a fixed peak-power constraint is given by [30]

$$\dot{C}(0) = \sup_{\substack{\mathbf{X} \neq \mathbf{0} \\ \|\mathbf{X}\|_\infty \leq \sqrt{K}}} \frac{D(p(\mathbf{Y}|\mathbf{X}) \| p(\mathbf{Y}|\mathbf{0}))}{\text{tr}(\mathbf{X}\mathbf{X}^*)}. \quad (107)$$

Using the expression for the Kullback–Liebler distance which can be obtained easily (cf. [17]), we obtain

$$\begin{aligned} \dot{C}(0) &= \sup_{\substack{\mathbf{X} \neq \mathbf{0} \\ \|\mathbf{X}\|_\infty \leq \sqrt{K}}} N_r \left(1 - \frac{\log \det(\mathbf{I}_T + \mathbf{X}\mathbf{X}^*)}{\text{tr}(\mathbf{X}\mathbf{X}^*)}\right) \\ &= \sup_{\substack{\mathbf{X} \neq \mathbf{0}, d \\ \text{tr}(\mathbf{X}\mathbf{X}^*) = d, \|\mathbf{X}\|_\infty \leq \sqrt{K}}} N_r \left(1 - \frac{\log \det(\mathbf{I}_T + \mathbf{X}\mathbf{X}^*)}{d}\right). \end{aligned} \quad (108)$$

Let the matrix $\mathbf{X}\mathbf{X}^*$ have eigenvalues $\{\lambda_i\}_{i=1}^T$. Then (109) can be upper-bounded as

$$\dot{C}(0) \leq \sup_{\substack{\{\lambda_i\}_{i=1}^T, d \\ \sum_i \lambda_i = d, d \leq KN_t T}} N_r \left(1 - \frac{\sum_i \log(1 + \lambda_i)}{d}\right) \quad (110)$$

$$= \sup_{d \leq KN_t T} N_r \left(1 - \frac{\log(1 + d)}{d}\right) \quad (111)$$

$$= N_r \left(1 - \frac{\log(1 + KN_t T)}{KN_t T}\right). \quad (112)$$

The expression in (111) is obtained by noting that since $-\sum_i \log(1 + \lambda_i)$ is a convex function of

$[\lambda_1 \ \lambda_2 \ \dots \ \lambda_T]^T$, the supremum in (110) is achieved at the extreme point $[d \ 0 \ \dots \ 0]^T$ by Lemma 2. Since $(1 - \log(1 + d)/d)$ is a monotonically increasing function of d , we obtain (112) by substituting the maximum value of d . The inequality in (110) is achieved with equality when \mathbf{X} is of unit rank, $\text{tr}(\mathbf{X}\mathbf{X}^*) = d$, and $\|\mathbf{X}\|_\infty \leq \sqrt{K}$. The supremum in (111) is achieved when $d = KN_t T$, and the unit rank \mathbf{X} satisfies both $\text{tr}(\mathbf{X}\mathbf{X}^*) = KN_t T$ as well as $\|\mathbf{X}\|_\infty \leq \sqrt{K}$ which in turn is true iff it is of the form given in (105).

To satisfy the average power constraint, set $P_1 = P/KN_t$. \square

D. Proof of Theorem 4

The results regarding generalized on–off signaling given in [6] are employed. In particular, note that Theorem 10 in [6] provides the $\frac{E_b}{N_0 \min}$ and \mathcal{S}_0 achieved by a generalized on–off signaling scheme. For convenience, that result is summarized here.

The generalized on–off signaling scheme has a P_0 mass at the all-zero matrix $\mathbf{0}_{T \times N_t}$. The input pdf conditioned on the input

$$\begin{aligned} \Delta(\bar{P}_Y \| P_{Y|X=0}) &= 1 + \sum_{i=1}^M \frac{P_i^2}{(1-P_0)^2} \frac{1}{|\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*|^{2N_r} |\mathbf{I} - (2\mathbf{I} - 2(\mathbf{I} + \mathbf{X}_i \mathbf{X}_i)^{-1})|} \\ &+ \sum_{\substack{i \\ j \neq i}} \frac{2P_i P_j}{(1-P_0)^2} \frac{1}{|\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*|^{N_r} |\mathbf{I} + \mathbf{X}_j \mathbf{X}_j^*|^{N_r} |(\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*)^{-1} + (\mathbf{I} + \mathbf{X}_j \mathbf{X}_j^*)^{-1} - \mathbf{I}|^{N_r}} \\ &- \sum_{i=1}^M \frac{P_i}{1-P_0} \frac{1}{|\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*|^{N_r} |\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*|^{-N_r}} \end{aligned} \quad (118)$$

being nonzero is denoted by \bar{P}_X , with distribution \bar{F}_X . With the input pdf conditioned on the all-zero matrix given by P^0 , the input pdf is

$$P_X = (P_0)P^0 + (1 - P_0)\bar{P}_X. \quad (113)$$

Denoting the pdf of the output conditioned on the input by $P_{Y|X}$, the output pdf corresponding to \bar{P}_X is given by

$$\bar{P}_Y = \int P_{Y|X=X} d\bar{F}_X(\mathbf{X}). \quad (114)$$

The wideband slope \mathcal{S}_0 achieved by generalized on-off signaling is

$$\mathcal{S}_0 = \frac{2}{T} \frac{\left(\mathbb{E}_{\bar{P}_X} [D(P_{Y|X} \| P_{Y|X=0})] \right)^2}{\Delta(\bar{P}_Y \| P_{Y|X=0})} \quad (115)$$

where $\Delta(\cdot, \|\cdot)$ denotes the Pearson's χ -divergence and is defined as

$$\Delta(\bar{P}_Y \| P_{Y|X=0}) \triangleq \mathbb{E}_{P_{Y|X=0}} \left[\left(\frac{\bar{P}_Y}{P_{Y|X=0}} - 1 \right)^2 \right]. \quad (116)$$

For the channel model under consideration in this paper, we have

$$\begin{aligned} \bar{P}_Y &= \sum_{i=1}^M \frac{P_i}{(1-P_0)} \frac{1}{\pi^{TN_r} |\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*|^{N_r}} \\ &\times \exp \left\{ -\text{tr} \left(\mathbf{Y}^* (\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*)^{-1} \mathbf{Y} \right) \right\} \\ P_{Y|X=0} &= \frac{\exp(-\text{tr}(\mathbf{Y}^* \mathbf{Y}))}{\pi^{TN_r}} \end{aligned}$$

and using the above expressions in (116), one obtains

$$\begin{aligned} \Delta(\bar{P}_Y \| P_{Y|X=0}) &= \mathbb{E}_{P_{Y|X=0}} \left[\sum_{i=1}^M \frac{P_i^2}{(1-P_0)^2} \frac{e^{2\text{tr}(\mathbf{Y}^* (\mathbf{I} - (\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*)^{-1}) \mathbf{Y})}}{|\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*|^{2N_r}} \right. \\ &+ 2 \sum_{\substack{i, j \\ i \neq j}} \frac{P_i P_j}{(1-P_0)^2} \frac{e^{\text{tr}(\mathbf{Y}^* (2\mathbf{I} - (\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*)^{-1} - (\mathbf{I} + \mathbf{X}_j \mathbf{X}_j^*)^{-1}) \mathbf{Y})}}{|\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*|^{N_r} |\mathbf{I} + \mathbf{X}_j \mathbf{X}_j^*|^{N_r}} \\ &\left. - 2 \sum_{i=1}^M \frac{P_i}{1-P_0} \frac{e^{\text{tr}(\mathbf{Y}^* (\mathbf{I} - (\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*)^{-1}) \mathbf{Y})}}{|\mathbf{I} + \mathbf{X}_i \mathbf{X}_i^*|^{N_r}} + 1 \right]. \quad (117) \end{aligned}$$

The above expression can be evaluated using the result from [38] that if \mathbf{z} is $\mathcal{CN}(\mathbf{0}, \mathbf{K})$ distributed, then $\mathbb{E}_{\mathbf{z}} [\exp(\mathbf{z}^* \mathbf{A} \mathbf{z})] = \{\det(\mathbf{I} - \mathbf{K} \mathbf{A})\}^{-1}$ if $\mathbf{I} - \mathbf{K} \mathbf{A}$ is positive definite. Otherwise, the expectation diverges. Hence (117) becomes (118) (shown

at the top of the page) if $\mathbf{I} - \mathbf{X}_i \mathbf{X}_i^* \mathbf{X}_j \mathbf{X}_j^*$ is positive definite $\forall i, j$, and ∞ otherwise. Simplification of (118) results in $\Delta(\bar{P}_Y \| P_{Y|X=0})$ given in Theorem 4. \square

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